# ON COMPLETION PROBLEMS FOR VARIOUS CLASSES OF $P$-MATRICES

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We dedicate this paper to Richard Brualdi with gratitude for all his contributions to linear algebra.

**Abstract.** A $P$-matrix is a real square matrix having every principal minor positive, and a Fischer matrix is a $P$-matrix that satisfies Fischer's inequality for all principal submatrices. In this paper, all patterns of positions for $n \times n$ matrices, $n \leq 4$, are classified as to whether or not every partial $\Pi$-matrix can be completed to a $\Pi$-matrix for $\Pi$ any of the classes positive $P$-, nonnegative $P$-, or Fischer matrices. Also, all symmetric patterns for $5 \times 5$ matrices are classified as to completion of partial Fischer matrices, and all but 2 such patterns are classified as positive $P$- or nonnegative $P$-completion. We also show that any pattern whose digraph contains a minimally chordal symmetric-Hamiltonian induced subdigraph does not have $\Pi$-completion for $\Pi$ any of the classes positive $P$-, nonnegative $P$-, Fischer matrices.

**Key words.** Matrix completion, partial matrix, $P$-matrix, nonnegative $P$-matrix, positive $P$-matrix, Fischer matrix, weakly sign-symmetric $P$-matrix

**AMS subject classifications.** 15A48, 05C50

**1. Introduction.** A *partial matrix* is a rectangular array in which some entries may be left unspecified, and a *completion* of a partial matrix is a particular choice of values for unspecified entries. Matrix completion problems have been explored for many classes of matrices. Applications of these problems arise in situations where some information is known but other information not available, and it is known that the full data matrix must have certain properties. Examples include geophysical problems [1], such as the analysis of seismic data, and computer engineering problems including data transmission, coding, decompression and image enhancement. Matrix completion problems also arise in optimization and in the study of Euclidean distance matrices [13].

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For $\alpha$ a subset of $\{1, \ldots, n\}$, the *principal submatrix* $A[\alpha]$ is obtained by deleting from $A$ all rows and columns whose indices are not in $\alpha$. A *principal minor* is the determinant of a principal submatrix. An $n \times n$ matrix is called a *P-matrix* ($P_0$-*matrix*) if all its principal minors are positive (nonnegative). Positive definite matrices are precisely the symmetric $P$-matrices. In this paper we study matrix completion problems for several classes of $P$-matrices, including the class of *positive (nonnegative) P-matrices*, i.e., those $P$-matrices whose entries are all positive (nonnegative), and *Fischer matrices*, i.e., $P$-matrices that satisfy the following condition (referred to as the *Fischer condition*) for every pair of disjoint subsets $\alpha$ and $\beta$ of $\{1, \ldots, n\}$:

$$\det A[\alpha] \det A[\beta] \geq \det A[\alpha \cup \beta]$$

Fischer matrices are a subclass of the *weakly sign symmetric P-matrices*, i.e., matrices $A = [a_{ij}]$ satisfying $a_{ij} a_{ji} \geq 0$ for all $i, j = 1, \ldots, n$.

Throughout, we let $\Pi$ denote one of the following properties: positive $P$, nonnegative $P$, nonnegative $P_0$, weakly sign symmetric $P$, or Fischer. By a $\Pi$-*matrix* we mean a matrix satisfying property $\Pi$. All of these classes of $\Pi$-matrices are *hereditary*, in the sense that any principal submatrix of a $\Pi$-matrix is a $\Pi$-matrix ($\Pi$ referring to the same type of matrix in both instances). Thus, a necessary condition for completion to a $\Pi$-matrix is that every fully specified principal submatrix is a $\Pi$-matrix. For $\Pi$ one of the properties weakly sign symmetric $P$ or Fischer, we define a *partial* $\Pi$-*matrix* to be a partial matrix in which every fully specified principal submatrix is a $\Pi$-matrix. If $\Pi$ is one of the properties positive $P$, nonnegative $P$, nonnegative $P_0$, in addition it is necessary that any specified entries be positive or nonnegative. So, for $\Pi$ one of the properties nonnegative $P$ or nonnegative $P_0$ (respectively, positive $P$), we define a *partial* $\Pi$-*matrix* to be a partial matrix in which every fully specified principal submatrix is a $\Pi$-matrix and every specified entry is nonnegative (respectively, positive).

A *pattern* for $n \times n$ matrices is a list of positions in an $n \times n$ matrix, that is, a subset of $\{1, \ldots, n\} \times \{1, \ldots, n\}$. A partial matrix *specifies a pattern* if its specified entries lie exactly in those positions listed in the pattern. A pattern is called *symmetric* if $(j, i)$ is in the pattern whenever $(i, j)$ is in the pattern; such patterns are also called positionally symmetric or combinatorially symmetric.

Graphs and digraphs are used to study patterns when all diagonal positions are present (we assume all diagonal entries are specified in every matrix in this paper). The digraph $G = (V_G, E_G)$ of a pattern with $n$ different indices has $n$ vertices labeled with the indices used and the arc $(i, j)$ for each position $(i, j)$ in the pattern. Alternatively, if the pattern is symmetric, it can also be modeled by a graph with undirected edges instead of directed arcs.

A pattern (or its (di)graph) *has $\Pi$-completion* if every partial $\Pi$-matrix which specifies the pattern can be completed to a $\Pi$-matrix. Otherwise the pattern (or the (di)graph) does not have $\Pi$-completion. The *matrix completion problem for $\Pi$-matrices* asks which patterns have $\Pi$-completion.

An induced subdigraph of $G$ is obtained from $G$ by deleting a set of vertices and every edge incident with a deleted vertex. Since an induced sub(di)graph $H$ of a

(di)graph $G$ corresponds to a principal submatrix of a matrix that specifies $G$, if $H$ does not have $\Pi$-completion, neither does $G$.

The underlying graph $G'$ of a digraph $G$ is the graph with the same vertex set and with edge set obtained by replacing each arc $(i, j)$ or pair of arcs $(i, j)$ and $(j, i)$ (if both are present) by the one edge $\{i, j\}$. Arc $(i, j)$ (or arcs $(i, j)$ and $(j, i)$ if both are present) of $G$ and edge $\{i, j\}$ of $G'$ are said to *correspond*.

A *path* in a digraph $G = (V_G, E_G)$ is sequence of vertices $v_1, v_2, \ldots, v_{k-1}, v_k \in V_G$ such that for $i = 1, \ldots, k - 1$, the arc $(v_i, v_{i+1}) \in E_G$. A *cycle* is a sequence of vertices $v_1, v_2, \ldots, v_{k-1}, v_k, v_1 \in V_G$ such that $v_1, \ldots, v_k$ is a path and also the arc $(v_k, v_1) \in E_G$. The definitions of path and cycle for a graph are analogous, with "arc" being replaced by "edge." A graph is connected if there is a path from any vertex to any other vertex (a graph of order 1 is connected); otherwise it is disconnected. A digraph is connected if its underlying graph is connected. A component of a (di)graph is a maximal connected sub(di)graph. A digraph is strongly connected if there is a path from any vertex to any other vertex. Clearly, a strongly connected digraph is connected, although the converse is false. In a graph, a chord of a cycle is an edge joining two non-consecutive vertices of the cycle. A graph $G$ is chordal if any cycle of length $> 3$ in $G$ has a chord. A cycle of a graph $G$ is a Hamiltonian cycle of $G$ if it contains all the vertices of $G$.

A (di)graph that contains all possible arcs between its vertices is called a clique. A cut-vertex of a connected (di)graph is a vertex that if removed, along with all incidental arcs, results in a disconnected (di)graph; more generally, a cut-vertex of a (di)graph is a vertex that is a cut-vertex of the component in which it is contained. If a connected (di)graph does not contain any cut-vertices, the (di)graph is nonseparable. A block is a maximal nonseparable sub(di)graph, and a (di)graph where all blocks are cliques is called block-clique (the term 1-chordal is also used to describe such a (di)graph).

For any arc $(i, j)$ in a digraph $G$, the reverse arc is $(j, i)$ (whether it is present in the digraph or not). An arc $(i, j)$ in $G$ is *symmetric in $G$* if its reverse arc is also in $G$; otherwise $(i, j)$ is *asymmetric in $G$*. A digraph $G$ is *symmetric* if every arc of $G$ is symmetric in G. Note that a pattern is symmetric if and only if its digraph is symmetric.

We will use the following properties of the matrix classes concerned in classifying patterns as to $\Pi$-completion:

- $\Pi$-*matrices are closed under permutation similarity.* Let $A$ be a $\Pi$-matrix. If $P$ is a permutation matrix, then $P^{-1}AP$ is a $\Pi$-matrix. This property allows us to classify an unlabeled graph or digraph of a pattern by examining a single labeling of its vertices.
- $\Pi$-*matrices are closed under positive diagonal multiplications.* Let $A$ be a $\Pi$-matrix. If $D$ is a positive diagonal matrix, then $DA$ is a $\Pi$-matrix. This allows us to assume the diagonal entries of $A$ are all equal to 1 when classifying the (di)graph of the pattern specified by $A$ as to $\Pi$-completion.

That these classes of matrices share these and other properties, and that all are subclasses of the class of $P$-matrices, explain the similarity in the method used to classify patterns with regard to completion for each class.

Naturally, the classes of positive $P$-matrices and nonnegative $P$-matrices have far more similarities. In fact, a pattern that has nonnegative $P$-completion necessarily has positive $P$-completion [9], and there is no known example of a pattern that has positive $P$-completion but does not have nonnegative $P$-completion.

An important distinction between Fischer and nonnegative $P$-matrices is that Fischer matrices allow negative entries. We will see in Section 3 that this causes the asymmetric 3-cycle to lack Fischer completion, whereas every $3 \times 3$ partial nonnegative $P$-matrix can be completed to a nonnegative $P$-matrix [5], [7]. In this regard, Fischer matrices behave like weakly sign symmetric $P$-matrices. The asymmetric 3-cycle also lacks weakly sign symmetric $P$-completion [3]. However, the Fischer condition is more demanding than being a weakly sign symmetric $P$-matrix, as evidenced by the fact that there are 5 order four digraphs that have weakly sign symmetric $P$-completion but do not have Fischer completion ([3] and Theorem 3.7).

Section 2 presents the classification of digraphs up through order four as to positive (nonnegative) $P$-completion, and additional material on minimally chordal symmetric-Hamiltonian digraphs. The classification of digraphs up through order four and of graphs through order five as to Fischer completion is given in Section 3.

A particularly productive similarity among the three classes (and many other related classes) that is very helpful in this classification, is the ability to complete a partial $\Pi$-matrix specifying a block-clique graph [5] and more generally, a digraph having every strongly connected induced subdigraph block-clique [7]. In order to classify all order four digraphs as to positive (nonnegative) $P$-completion, we identify a (di)graph as in [6], pp. 215-230, where $q$ is the number of arcs (edges) and $n$ is the diagram number.

LEMMA 1.1. *For the following order four digraphs, every strongly connected induced subdigraph is block-clique. Thus they have $\Pi$-completion by [5] and [7]:*

$q = 0;$
$q = 1;$
$q = 2 \qquad n = 1\text{-}5;$
$q = 3 \qquad n = 1\text{-}11, 13;$
$q = 4 \qquad n = 1\text{-}12, 14, 15, 17\text{-}19, 21\text{-}23, 25\text{-}27;$
$q = 5 \qquad n = 1\text{-}5, 8\text{-}10, 14\text{-}17, 21\text{-}24, 26\text{-}29, 31, 33, 34, 36, 37;$
$q = 6 \qquad n = 1\text{-}3, 5, 6, 8, 13, 15, 17, 19, 23, 26\text{-}27, 32, 35, 38\text{-}40, 43, 46;$
$q = 7 \qquad n = 4, 5, 9, 14, 24, 29, 34, 36;$
$q = 8 \qquad n = 1, 10, 12, 18;$
$q = 9 \qquad n = 8, 11;$
$q = 12.$

LEMMA 1.2. *Every component of each of the following order five graphs is block-clique. Thus they have $\Pi$-completion by [5]:*

$q = 0;$
$q = 1;$
$q = 2$        $n = 1,2;$
$q = 3$        $n = 1\text{-}4;$
$q = 4$        $n = 2\text{-}6;$
$q = 5$        $n = 2, 4\text{-}5;$
$q = 6$        $n = 1, 3;$
$q = 7$        $n = 3;$
$q = 10.$

**2. Positive (nonnegative) $P$-completion.** The class of nonnegative $P$-matrices includes the classes of inverse $M$-matrices and nonsingular totally nonnegative matrices. Fallat, et al [5] initiated the study of the positive (nonnegative) $P$-matrix completion problem and developed techniques that allow completion of any partial positive (nonnegative) $P$-matrix specifying any one of a large collection of graphs. Most of their results are based on the assumption that the patterns are symmetric, but extend naturally to a large number of patterns that are not symmetric. In this section we use these results, results of [7] and [12], and new results to classify digraphs of order up through 4 as to whether every partial positive (nonnegative) $P$-matrix having the pattern of specified entries described by the digraph can be completed to a positive (nonnegative) $P$-matrix. We also extend work done in [4] to obtain a family of digraphs that lack positive and nonnegative $P$-completion.

We begin with a survey of previous results that we use.

LEMMA 2.1.   *[8] If a digraph has nonnegative $P_0$-completion then it also has nonnegative $P$-completion.*

LEMMA 2.2.   *[9] If a digraph has nonnegative $P$-completion then it also has positive $P$-completion.*

LEMMA 2.3.   *[5], [7] Any $3 \times 3$ partial positive (nonnegative) $P$-matrix can be completed to a positive (nonnegative) $P$-matrix, i.e., every order $3$ digraph has positive (nonnegative) $P$-completion.*

As a consequence of the last result we have the following corollary.

COROLLARY 2.4.   *Let $G$ be a digraph that has positive (nonnegative) $P$-completion. Let $H$ be a digraph obtained from $G$ by deleting one arc $(u, v)$ such that $u$ and $v$ are contained in at most one clique of order $3$ in $G$. Then $H$ has positive (nonnegative) $P$-completion.*

*Equivalently, if $G$ is a digraph obtained from a digraph $H$ by adding one arc $(u, v)$ such that $u$ and $v$ are contained in at most one clique of order $3$ in $G$, and $H$ does not have positive (nonnegative) $P$-completion, then $G$ does not have positive (nonnegative) $P$-completion.*

*Proof.* We establish the first statement. Let $A$ be a partial positive (nonnegative) $P$-matrix specifying $H$. Choose a value for the unspecified $u, v$-entry of $A$ to obtain a partial positive (nonnegative) $P$-matrix $B$ specifying $G$ as follows: If $u$ and $v$ are not in any clique of order 3, set the $u, v$-entry of $B$ to 0 (or a sufficiently small positive value). If the subdigraph induced by $\{u, v, w\}$ is a clique in $G$, choose a value $c$ for the $u, v$-entry that completes $A(\{u, v, w\})$ to a positive (nonnegative) $P$-matrix (such

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a $c$ is guaranteed to exist by Lemma 2.3). Then, since $G$ has positive (nonnegative) $P$-completion, we can complete $B$ to a positive (nonnegative) $P$-matrix $C$, which also completes $A$. Thus $H$ has positive (nonnegative) $P$-completion. ☐

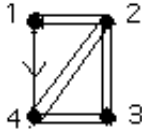LEMMA 2.5. *The digraphs $q = 9, n = 1, 2$ (see Figure 2.1 for $q = 9, n = 1$) have nonnegative $P$-completion.*

*Proof.* The result is established by providing an appropriate completion for any partial nonnegative $P$-matrix $A$ and verifying that the completion is indeed a $P$-matrix. Let the partial positive $P$-matrix be $A = \begin{bmatrix} 1 & a_{12} & x & a_{14} \\ a_{21} & 1 & a_{23} & a_{24} \\ y & a_{32} & 1 & a_{34} \\ z & a_{42} & a_{43} & 1 \end{bmatrix}$. The completion chosen depends on certain entries of $A$. Here we provide appropriate completions for all possible cases. It is immediate that the given completions are nonnegative $P$-matrices. To finish the proof, it must be verified that each completed matrix is a $P$-matrix, which requires checking that all principal minors are positive. This is tedious but not difficult; the algebraic details are omitted.

- $a_{21} = 0$: Set all unspecified entries to 0.
- $a_{21} \neq 0$ and $(a_{42} = 0$ or $(a_{14}a_{42}a_{21} \geq 1$ and $a_{32} - a_{34}a_{42} > 0))$: Set $z = 0$, $y = 0$, and choose $x$ sufficiently large to ensure all principal minors are positive.
- $a_{21} \neq 0$ and $a_{42} \neq 0$ and $(a_{14}a_{42}a_{21} \geq 1$ and $a_{32} - a_{34}a_{42} \leq 0))$: Set $z = 0$, $y = 0$, and $x = a_{12}a_{23}$.
- $a_{21} \neq 0$ and $a_{42} \neq 0$ and $a_{14}a_{42}a_{21} < 1$: Set $z = a_{42}a_{21}$, $y = 0$, and choose $x$ sufficiently large to ensure all principal minors are positive.

The digraph $q = 9, n = 2$ is obtained from $q = 9, n = 1$ by reversing the direction of all arcs. Thus any partial nonnegative $P$-matrix $A$ specifying $q = 9, n = 2$ may be transposed to obtain a partial nonnegative $P$-matrix specifying $q = 9, n = 1$, completed to a nonnegative $P$-matrix, and transposed to obtain a completion of $A$. ☐

We now have enough information to classify all order four digraphs.

THEOREM 2.6. *An order four digraph has positive (nonnegative) $P$-completion if and only if it is not one of the digraphs listed below.*

$q = 6 \quad n = 45;$

$q = 7 \quad n = 30, 32, 33, 35, 38;$

$q = 8 \quad n = 16, 17, 19, 20, 22\text{-}26;$

$q = 9 \quad n = 3\text{-}7, 9, 10, 12, 13;$

$q = 10 \quad n = 1\text{-}5;$

$q = 11.$

*Proof.*

## Noncompletion

In Proposition 3.2 (a) of [12], the partial positive $P$-matrix

$$\begin{bmatrix} 1 & 1 & x_{13} & x_{14} \\ x_{21} & 1 & 1 & 0.2 \\ 20 & x_{32} & 1 & 1 \\ 40 & x_{42} & x_{43} & 1 \end{bmatrix}$$ is used to show that $q = 6, n = 45$ (numbered 1, 2, 3,

4 counterclockwise from the lower right) does not have positive $P$-completion. All of the remaining digraphs except $q = 9, n = 3; q = 10, n = 1$; and $q = 11$ can be obtained from $q = 6, n = 45$ by adding one arc at a time without completing more than one order 3 digraph. Thus they do not have positive $P$-completion by Corollary 2.4. In Proposition 3.4 ($U = 3$) of [12], the partial positive $P$-matrix

$$\begin{bmatrix} 1 & 1 & 5 & 7 \\ 0.99 & 1 & x_{23} & 7.2 \\ 0.167 & x_{32} & 1 & 1 \\ x_{41} & 0.138 & 0.5 & 1 \end{bmatrix}$$ is used to show that $q = 9, n = 3$ (numbered 1, 3, 4,

2 clockwise from the upper right) does not have positive $P$-completion. In Lemma

2.3 of [5], the partial positive $P$-matrix $\begin{bmatrix} 1 & 1 & 0.01 & x_{14} \\ 0.45 & 1 & 1 & 0.01 \\ 0.01 & 0.45 & 1 & 1 \\ x_{41} & 0.01 & 0.45 & 1 \end{bmatrix}$ is used to show

that $q = 10, n = 1$ (numbered 1, 3, 4, 2 clockwise from the upper left) does not have positive $P$-completion. The digraph $q = 11$ can be obtained from $q = 10, n = 1$ by adding one arc without completing any principal submatrices, so $q = 11$ does not have positive $P$-completion by Corollary 2.4. These digraphs do not have nonnegative $P$-completion by Lemma 2.2.

## Completion

Any digraph not listed above has completion because it fits into at least one of the following categories.

- Digraphs that have nonnegative $P$-completion because each strongly connected induced subdigraph is block-clique, see Lemma 1.1.
- Digraphs that have nonnegative $P_0$-completion [2], and thus have nonnegative $P$-completion by Lemma 2.1.
- Every partial nonnegative $P$-matrix the digraph of whose specified entries is a symmetric 4-cycle can be completed to a nonnegative $P$-matrix, [5], [7].
- Digraphs that are obtained by deleting one arc at a time from a digraph listed above (without breaking more than one 3-clique), see Corollary 2.4.

Finally, all these digraphs have positive $P$-completion by Lemma 2.2. □

All the order 5 graphs listed in Lemma 1.2 have positive (nonnegative) $P$-comple-

tion. Any order 5 graph that contains the double triangle as an induced subgraph does not have positive (nonnegative) $P$-completion [5]. This covers all order 5 graphs except the two shown in Figure 2.2.
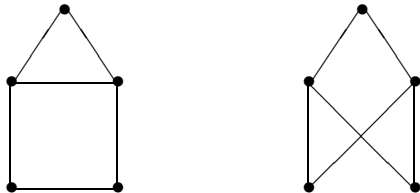
FIG. 2.2. *The two order 5 graphs not classified with respect to positive (nonnegative) P-completion*

Next we provide an infinite family of digraphs that have neither positive $P$-completion nor nonnegative $P$-completion. An example of a digraph in this family is shown in Figure 2.3.

In Example 2 of [11], two $3 \times 3$ principal submatrices that include the same unspecified entry are used to give contradictory requirements, showing that a $4 \times 4$ partial $P$-matrix with exactly one unspecified entry need not have completion to a $P$-matrix. In [4] this example was generalized to show that the family of minimally chordal symmetric-Hamiltonian digraphs lack $P$-completion.

The graph $G$ is a *minimally chordal Hamiltonian graph* if
1. $G$ has a Hamiltonian cycle $H$.
2. $G$ is chordal.
3. If any non-empty set $S$ of chords of $H$ is removed from $G$, the resulting graph is not chordal.

It is established in [4] that the Hamilton cycle of a minimally chordal Hamiltonian graph is unique.

The digraph $G$ is a *minimally chordal symmetric-Hamiltonian digraph* if
1. The underlying graph $G'$ of $G$ is a minimally chordal Hamiltonian graph.
2. Each arc corresponding to an edge of the unique Hamiltonian cycle of $G'$ is symmetric in $G$.
3. Each arc corresponding to a chord of the unique Hamiltonian cycle of $G'$ is asymmetric in $G$.

For a minimally chordal symmetric-Hamiltonian digraph $G$, we can specify a partial positive $P$-matrix and use the principal submatrices $A_1, A_2, A_3, A_4$ from [4], all of whose specified entries are close to 1, to "squeeze" (force close to 1) unspecified entries corresponding to the reverses of chord arcs of $G$. But we can also use $A_5$ below to produce a contradictory requirement on the value of one entry. Thus such a partial positive $P$-matrix cannot be completed to a $P$-matrix.

LEMMA 2.7. *If the matrix* $A_5 = \begin{bmatrix} 1 & 0.1 & 1 \\ 0.4 & 1 & 0.9 \\ 0.8 & y & 1 \end{bmatrix}$ *is a P-matrix, then* $y <$
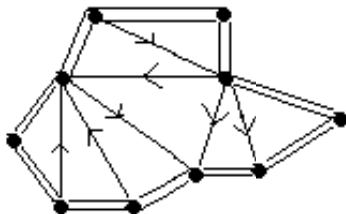
FIG. 2.3. *A minimally chordal symmetric-Hamiltonian digraph*

0.464.

*Proof.* $0 < \det A_5 = 0.232 - 0.5y$. □

By using the matrix $A_5$ given in Lemma 2.7 above instead of the matrix $A_5$ in [4], the proofs in [4] establish the following results.

THEOREM 2.8. *For any pattern $Q$ that contains all diagonal positions and whose digraph is a minimally chordal symmetric-Hamiltonian digraph with at least 4 vertices, there exists a partial positive $P$-matrix that cannot be completed to a $P$-matrix.*

COROLLARY 2.9. *Let $G$ be a minimally chordal symmetric-Hamiltonian digraph. Then there exists a partial positive (nonnegative) $P$-matrix specifying $G$ that cannot be completed to a positive (nonnegative) $P$-matrix. In other words, $G$ does not have positive (nonnegative) $P$-completion.*

COROLLARY 2.10. *Let $Q$ be pattern and let $G$ be its digraph. If $G$ contains a subdigraph $F$ that is a minimally chordal symmetric-Hamiltonian digraph containing at least four vertices, and such that the subdigraph of $G$ induced by the vertices of $F$ does not contain any arc symmetric in $G$ except those in the Hamiltonian cycle of $F$, then $Q$ does not have positive (nonnegative) $P$-completion.*

Note that it was shown in [3] that any graph that contains a 3-cycle whose induced subgraph is not a clique does not have (weakly) sign symmetric $P$-completion, so any minimally chordal symmetric-Hamiltonian digraph does not have (weakly) sign symmetric $P$-completion, and thus it follows from Corollary 3.2 below, does not have Fischer completion.

**3. Fischer matrices.** Fischer matrices merit study because they contain many interesting classes of matrices, for example, positive definite matrices, triangular matrices with positive diagonal, $M$-matrices, inverse $M$-matrices, and nonsingular totally nonnegative matrices [5]. This section contains a complete classification of the patterns for $4 \times 4$ matrices into those having Fischer completion and those not having Fischer completion, and similar classification of symmetric patterns for $5 \times 5$ matrices. The classification of patterns is carried out by analysis of the corresponding digraphs (graphs) with four (five) or fewer vertices.

The results in the following lemma are well-known and easily proved by applying the Fischer condition to $2 \times 2$ principal submatrices.

LEMMA 3.1. *Every Fischer matrix is a weakly sign symmetric $P$-matrix. Every $2 \times 2$ weakly sign-symmetric $P$-matrix is a Fischer matrix.*

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Corollary 3.2. *A digraph that does not have weakly sign symmetric P-completion and that does not contain any 3-clique does not have Fischer completion.*

A partial Fischer matrix specifying any one of the order two digraphs may be completed to a Fischer matrix by replacing each unspecified entry with a zero. The following lemma follows from [3] and Corollary 3.2 (for noncompletion) and [7] (every strongly connected induced subdigraph block-clique implies completion).

Lemma 3.3. *A pattern for $3 \times 3$ matrices has Fischer completion if and only if its digraph does not contain a 3-cycle or is complete.*

Corollary 3.4. *If a digraph contains a 3-cycle that is not symmetric as an induced subdigraph, the pattern does not have Fischer completion. The order four digraphs having such a 3-cycle are:*

$q = 3 \qquad n = 12;$
$q = 4 \qquad n = 13, 20, 24;$
$q = 5 \qquad n = 6, 11\text{-}13, 18\text{-}20, 25, 30, 32, 35, 38;$
$q = 6 \qquad n = 9\text{-}12, 14, 16, 18, 20\text{-}22, 24, 25, 28\text{-}31$
$\qquad\qquad n = 33, 34, 36, 37, 41, 42, 44, 45, 47, 48;$
$q = 7 \qquad n = 1, 3, 6\text{-}8, 10\text{-}13, 15\text{-}23, 25\text{-}28, 30\text{-}33, 35, 37, 38;$
$q = 8 \qquad n = 3\text{-}9, 11, 13\text{-}17, 19\text{-}27;$
$q = 9 \qquad n = 1\text{-}7, 9, 10, 12, 13;$
$q = 10 \qquad n = 2\text{-}5;$
$q = 11.$

The proof of Corollary 2.4 can easily be modified to prove the next result.

Lemma 3.5. *Let $G$ be a digraph that has Fischer completion. Let $H$ be a digraph obtained from $G$ by deleting one arc $(u, v)$ such that $u$ and $v$ are not contained in a clique of order 3 in $G$. Then $H$ has Fischer completion.*

*Equivalently, if $G$ is a digraph obtained from digraph $H$ by adding one arc $(u, v)$ such that $u$ and $v$ are not contained in a clique of order 3 in $G$, and $H$ does not have Fischer completion, then $G$ does not have Fischer completion.*

Lemma 3.6. *The order four digraph $q = 4$, $n = 16$ (the asymmetric 4-cycle) does not have Fischer completion.*

*Proof.* The partial matrix $A = \begin{bmatrix} 1 & 10 & x_{13} & x_{14} \\ x_{21} & 1 & -10 & x_{24} \\ x_{31} & x_{32} & 1 & 10 \\ 10 & x_{42} & x_{43} & 1 \end{bmatrix}$ is a partial Fischer matrix specifying the digraph $q = 4$, $n = 16$. We will verify that $A$ does not have a completion to a Fischer matrix by assuming that $A$ has Fischer completion and deriving a contradiction. We first consider the $2 \times 2$ principal minors of $A$, where, combined with the guarantee of weak sign-symmetry, we obtain the restrictions in Table 3.1. Using these restrictions on the unspecified entries $x_{ij}$, we obtain further restrictions from $3 \times 3$ principal minors as shown in Table 3.2.

Now consider the Fischer condition with $\alpha = \{2, 3, 4\}$, $\beta = \{1\}$:

$$\det A \leq \det A[2, 3, 4] \det A[1]$$

Using this condition and the restrictions from Table 3.1 and Table 3.2, we obtain the

| $\alpha$ | $\mathrm{Det}A(\alpha)$ | Restriction |
|---|---|---|
| {1,2} | $1 - 10x_{21}$ | $0 \leq 10x_{21} < 1$ |
| {1,4} | $1 - 10x_{14}$ | $0 \leq 10x_{14} < 1$ |
| {3,4} | $1 - 10x_{43}$ | $0 \leq 10x_{43} < 1$ |
| {2,3} | $1 + 10x_{32}$ | $-1 < 10x_{32} \leq 0$ |
| {1,3} | $1 - x_{13}x_{31}$ | $0 \leq x_{13}x_{31} < 1$ |
| {2,4} | $1 - x_{24}x_{42}$ | $0 \leq x_{24}x_{42} < 1$ |

<div align="center">TABLE 3.1</div>

| $\alpha$ | $\det A(\alpha)$ | Restriction |
|---|---|---|
| {1,2,3} | $1 - 10x_{21} - 100x_{31} - x_{13}x_{31} + 10x_{32} + x_{13}x_{21}x_{32}$ | $1 - 100x_{31} + x_{13}x_{21}x_{32} > 0$ |
| {1,2,4} | $1 - 10x_{14} - 10x_{21} + 100x_{24} + x_{14}x_{21}x_{42} - x_{24}x_{42}$ | $1 + 100x_{24} + x_{14}x_{21}x_{42} > 0$ |
| {1,3,4} | $1 + 100x_{13} - 10x_{14} - x_{13}x_{31} - 10x_{43} + x_{14}x_{31}x_{43}$ | $1 + 100x_{13} + x_{14}x_{31}x_{43} > 0$ |

<div align="center">TABLE 3.2</div>

inequality

$$(3.1) \qquad 10x_{13}x_{24}x_{32} + 10x_{13}x_{21}x_{42} - 10x_{14}x_{31}x_{42} + 10x_{24}x_{31}x_{43} \geq 9994$$

Further, since $10x_{32}$, $10x_{21}$, $10x_{14}$, and $10x_{43}$ are all less than 1 in absolute value, the triangle inequality gives

$$|x_{13}x_{24}| + |x_{13}x_{42}| + |x_{31}x_{24}| + |x_{31}x_{42}| \geq 9994$$

To satisfy the inequality, at least one of the four remaining terms must have absolute value greater than 2000. To meet this requirement, at least one of the $x_{ij}$ components of at least one term must have absolute value greater than 40. Thus, letting some $|x_{ij}| > 40$, we must consider four cases (each with two subcases, denoted (a) and (b)) based on Inequality 3.1: **(i)** $|x_{13}| \geq 40$, **(ii)** $|x_{31}| \geq 40$, **(iii)** $|x_{24}| \geq 40$, and **(iv)** $|x_{42}| \geq 40$.

**Case (i):** $|x_{13}| \geq 40$.

(a): $x_{13} \geq 40$. Using the Fischer condition with $\alpha = \{3,4\}$, $\beta = \{1\}$, and the fact that $\det A[1] = 1$, we obtain the equation $100x_{13} - 10x_{14} - x_{13}x_{31} + x_{14}x_{31}x_{43} \leq 0$. Substituting into the left side and rearranging yields $100x_{13} + x_{14}x_{31}x_{43} \leq 2$. Now, because $x_{13} \geq 40$, $100x_{13} \geq 4000$. Further, $x_{14}x_{31}x_{43} \geq 0$ based on the restrictions from Table 3.1. This inequality, therefore, implies $4000 \leq 2$.

The remaining cases and subcases are similar. In each case, examine the following inequalities:

**Case(i)** (b) $x_{13} \leq -40$. Consider the (necessarily positive) determinant of the principal submatrix $A[1, 3, 4]$.

**Case (ii):** $|x_{31}| \geq 40$.

(a) $x_{31} \geq 40$. Consider the determinant of the principal submatrix $A[1, 2, 3]$.

(b) $x_{31} \leq -40$. Consider the Fischer condition with $\alpha = \{1, 3\}$, $\beta = \{2\}$.

**Case (iii):** $|x_{24}| \geq 40$.

(a) $x_{24} \geq 40$. Consider the Fischer condition with $\alpha = \{1, 4\}$, $\beta = \{2\}$.

(b) $x_{24} \leq -40$. Consider the determinant of the principal submatrix $A[1, 2, 4]$.

**Case (iv):** $|x_{42}| \geq 40$.

(a) $x_{42} \geq 40$. Consider the determinant of the principal submatrix $A[2, 3, 4]$.

(b) $x_{42} \leq -40$. Consider the Fischer condition with $\alpha = \{2, 4\}$, $\beta = \{3\}$.

Thus the asymmetric 4-cycle does not have Fischer completion. □

We are now ready to classify $4 \times 4$ digraphs as to Fischer completion.

THEOREM 3.7. *An order four digraph has Fischer completion if and only if every strongly connected induced subdigraph is block-clique (see list in Lemma 1.1).*

*Proof.* It is necessary only to establish noncompletion of the digraphs not listed in Lemma 1.1. Any such digraph falls into one of the following categories.

- The digraphs listed in Corollary 3.4, which contain a 3-cycle $v_1, v_2, v_3, v_1$ such that the subdigraph induced by $\{v_1, v_2, v_3\}$ is not a clique.
- The digraphs, $q = 4$, $n = 16$; $q = 5$, $n = 7$; $q = 6$, $n = 4, 7$; $q = 7$, $n = 2$, which can be created from the asymmetric 4-cycle ($q = 4$, $n = 16$) by adding 0, 1, 2, or 3 arcs without creating any 3-cliques, and so do not have Fischer completion by Lemmas 3.6 and 3.5.
- The symmetric 4-cycle ($q = 8$, $n = 2$) and the double triangle ($q = 10$, $n = 1$), which do not have Fischer completion by Example 3.3 in [5] and Lemma 2.3 in [5], respectively.

□

We now turn our attention to a complete classification of symmetric patterns for $5 \times 5$ matrices into those having Fischer completion and those not having Fischer completion. Symmetric patterns up through size 4 have already been classified. The classification of patterns is completed by analysis of the corresponding graphs with five vertices. As usual, all graphs are identified as in [6].

THEOREM 3.8. *An order five graph has completion if and only if each of its components is block-clique.*

*Proof.* It is necessary only to establish noncompletion of the order five graphs not listed in Lemma 1.2. Any such graph falls into one of the following categories.

- The graphs $q = 4$, $n = 1$; $q = 5$, $n = 1,3$; $q = 6$, $n = 2, 4, 5$-6; $q = 7$, $n = 1,2,4$; $q = 8$, $n = 1, 2$; $q = 9$, each of which contains a 4-cycle or a double triangle as an induced subgraph, so the graph does not have Fischer completion.
- The (symmetric) 5-cycle ($q = 5$, $n = 6$), does not have Fischer completion by Corollary 3.2 (since in [3] it is shown that the symmetric 5-cycle does not have weakly sign-symmetric $P$-completion).

□

Any order six graph that is block-clique has Fischer completion. Any order six graph that contains the double triangle, (symmetric) 4-cycle, or (symmetric) 5-cycle

as an induced subgraph does not have Fischer completion. This covers all order six graphs except the (symmetric) 6-cycle, which has not been classified. Note that all (di)graphs that have been shown to have Fischer completion have the property that every strongly connected induced subdigraph is block-clique. This raises the following question.

QUESTION 3.9. *Does every digraph having Fischer completion have the property that every strongly connected induced subdigraph is block-clique?*

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