MINIMUM RANK OF A TREE OVER AN ARBITRARY FIELD*

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Abstract. For a field F and graph G of order n, the minimum rank of G over F is defined to be the smallest possible rank over all symmetric matrices $A \in F^{n\times n}$ whose (i, j)th entry (for $i \neq j$) is nonzero whenever $\{i, j\}$ is an edge in G and is zero otherwise. We show that the minimum rank of a tree is independent of the field.

Key words. minimum rank, tree, graph, field, path, symmetric matrix.

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1. Introduction.

The *minimum rank problem* is the problem of finding the smallest rank of a matrix in the set of symmetric matrices having the zero-nonzero pattern of off-diagonal entries described by a given (simple undirected) graph. This problem has received considerable attention recently (see [1], [2], [3], [4], [5], [7], [9], [10], [11], [13], [14], [15]). Originally the minimum rank problem was studied over the real numbers, where it is equivalent to the question of maximum multiplicity of an eigenvalue of the same family of matrices. The study of minimum rank was expanded to arbitrary fields in [4] and [5].

In this paper, a *graph* $G = (V(G), E(G))$ is a simple undirected graph, i.e. a set $V(G)$ of *vertices* with a set $E(G)$ of two-element subsets of vertices called *edges*. The *order* of a graph G is the number of vertices in $V(G)$ and is denoted by $|G|$. A *connected* graph is a graph that has a path between any two vertices. A *tree* is a connected graph with no cycles.

Define $S_n(F)$ to be the set of all symmetric $n \times n$ matrices over a field F. The graph of $A \in S_n(F)$, denoted $\mathcal{G}(A)$, is the graph G with vertices $\{1, ..., n\}$ and edges $\{\{i, j\} \mid a_{ij} \neq 0 \text{ and } i \neq j\}$. The diagonal entries of A have no bearing on the structure of G. Define

$$\mathcal{S}^F(G) = \{A \in S_n(F) \mid \mathcal{G}(A) = G\}.$$

Define the *minimum rank* of G over F as

$$\mathrm{mr}^F(G) = \min\{\mathrm{rank}(A) \mid A \in \mathcal{S}^F(G)\}.$$

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In 1996 Nylen gave a method for computation of minimum rank for a tree, subsequently improved by Johnson and Leal-Duarte [11], Johnson and Saiago [14], and others. Convenient algorithms for computation of the minimum rank (over the reals) of a tree (by computation of the graph parameter Δ, cf. Section 2) are available, e.g., in [12]. In Section 2 we show that the minimum rank of a tree is independent of field.

We will need some additional terminology and notation to prove the main result. Given $R \subseteq V(G)$, the subgraph of G *induced* by R is $G[R] = (R, E(G[R]))$ where $E(G[R]) = \{\{i,j\} \in E(G) \mid i,j \in R\}$. The induced subgraph $G[V(G) \setminus R]$ will be denoted by $G(R)$. Let A be an $n \times n$ matrix and $R \subseteq \{1, 2, ..., n\}$. Let $A(R)$ denote the principal submatrix of A obtained by deleting the rows and columns in R. By a slight abuse of notation, if $A \in \mathcal{S}^F(G)$ and $R \subseteq \{1, \ldots, n\}$, then $A(R) \in \mathcal{S}^F(G(R))$ (technically, the rows and columns of the matrix should be indexed by R, but this distinction will be ignored). We use the following notation: P_n denotes the path on n vertices. K_n denotes the complete graph on n vertices.

The limited progress on the (real) minimum rank problem for graphs that are not trees has come mainly in two ways, by deleting cut-vertices (as is done in the tree algorithms), and by characterizing graphs having relatively extreme minimal rank. The connected graphs having minimum rank 1 are exactly the complete graphs K_n (and this is clearly independent of field). Of course, only connected graphs need be considered to compute minimum rank, as the minimum rank of a graph is the sum of the minimum ranks of its connected components.

Since a singular matrix can be obtained by adjusting diagonal entries to make every row sum equal to zero, it is clear that for any field F and graph G, $\operatorname{mr}^F(G) \leq |G| - 1$. For any field F, if $A \in \mathcal{S}^F(P_n)$ and P_n has the standard vertex labeling so A is tridiagonal, then the matrix obtained from A be deleting column 1 and row n is invertible. Hence $\operatorname{mr}^F(P_n) = n - 1$. In the study of minimum rank of a graph over the real numbers, it is well-known that $\operatorname{mr}^{\mathbb{R}}(G) = |G| - 1$ implies G is a path. This result is obtained as a consequence of the following theorem of Fiedler.

Theorem 1.1. [8] (Fiedler's Tridiagonal Matrix Theorem) *If A is a real symmetric $n \times n$ matrix such that for all real diagonal matrices D, $\operatorname{rank}(A+D) \geq n-1$, then A is irreducible and there is a permutation matrix P such that P^TAP is tridiagonal.*

Fiedler's proof of his Tridiagonal Matrix Theorem relies on continuity properties of the real numbers, but a combinatorial proof valid over any infinite field was subsequently given by Rheinboldt and Shepherd [16]. More recently, Bento and Leal-Duarte [6] established Fiedler's Theorem for all fields except for a few 5×5 matrices over $\mathbb{Z}_3$. Since the matrices involved have all off-diagonal entries nonzero, these matrices are in $\mathcal{S}^{\mathbb{Z}_3}(K_n)$, so their exceptional behavior in regards to Fielder's Tridiagonal Matrix Theorem does not affect the application of the theorem to minimum rank problems (since Fiedler's Theorem is not needed for K_n). Thus, Fiedler, Rheinboldt, Shepherd, Bento and Leal-Duarte have established that

$$\operatorname{mr}^F(G) = |G| - 1 \text{ if and only if } G = P_{|G|}.$$

In [4] and [5] graphs having minimum rank 2 over a field F are characterized by a short list of forbidden induced subgraphs, and in [7] is is shown that for a finite field, the graphs having minimum rank $\leq k$ can always be characterized by a finite set of forbidden induced subgraphs. In [13] all graphs having minimum rank $|G| - 2$ over a field F are characterized.

2. Trees. In this section we show that the minimum rank of a tree is the same for all fields. For any tree T, the *path cover number* $P(T)$ is the minimum number of vertex-disjoint induced paths needed to cover all vertices of T and

$$\Delta(T) = \max\{p - q \mid \text{there is a set of } q \text{ vertices whose deletion leaves } p \text{ paths}\}.$$

Eigenvalue interlacing is used in [11] to prove that $\text{mr}^{\mathbb{R}}(T) \leq |T| - \Delta(T)$ for any tree T. Unfortunately, this method is not meaningful for fields other than the reals, since the geometric and algebraic multiplicities of eigenvalues of symmetric matrices are no longer necessarily equal; for example, the matrix $\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \in \mathbb{C}^{2\times 2}$ has eigenvalue 0 with algebraic multiplicity 2 and geometric multiplicity 1, and eigenvalue interlacing is not valid for this matrix. However, the result $\text{mr}^F(T) \leq |T| - \Delta(T)$ remains valid and can be proved by examining the ranks of the matrices involved in the interlacing technique. A similar technique was used in [1].

Proposition 2.1. *For any field F and tree T, $\text{mr}^F(T) \leq |T| - \Delta(T)$.*

Proof. Let $n = |T|$. Recall that $\Delta(T) = \max(p - q)$ such that there exist q vertices of T whose deletion leaves p paths. Let Q be a set of q such vertices, where $G(Q)$ is p paths $P^{(1)}, P^{(2)}, \ldots, P^{(p)}$. Let n_i be the number of vertices in path $P^{(i)}$. For each $P^{(i)}$, choose an $n_i \times n_i$ matrix $A_i \in \mathcal{S}^F(P^{(i)})$ of rank $n_i - 1$. Embed each A_i in an $n \times n$ matrix A so that each A_i is the principal submatrix corresponding to the vertices of $P^{(i)}$. Add nonzero entries (of arbitrary value) symmetrically to the rows and columns of the q vertices deleted, in locations corresponding to existing edges of T, so that the resulting matrix is in $\mathcal{S}^F(T)$. Then

$$\text{rank}(A(Q)) = \sum_{i=1}^{p} \text{rank}(A_i) = \sum_{i=1}^{p} (n_i - 1).$$

Adding each row/column can increase the rank by at most 2. So

$$\text{rank}(A) \leq \sum_{i=1}^{p} (n_i - 1) + 2q = \sum_{i=1}^{p} n_i + q + (q - p) = n - \Delta(T).$$

Thus $\text{mr}^F(T) \leq n - \Delta(T)$. □

Let T be a tree. The proof of $mr^{\mathbb{R}}(T) \geq |T| - P(T)$ in the Theorem in [11] remains valid over an arbitrary field, i.e., for any field F, $mr^F(T) \geq |T| - P(T)$. Since $\Delta(T)$ and $P(T)$ are determined by T independent of the field, and since for $F = \mathbb{R}$, $\Delta(T) = |T| - \text{mr}^{\mathbb{R}}(T) = P(T)$ [11], we have established the following theorem.

Theorem 2.2. *For any field F and tree T, $mr^F(T) = |T| - \Delta(T) = |T| - P(T)$. In particular, for a tree, minimum rank is independent of field.*

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