Morse Decompositions, Attractors and Chain Recurrence

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Abstract

The global behavior of a dynamical system can be described by its Morse decompositions or its attractor and repeller configurations. There is a close relation between these two approaches and also with (maximal) chain recurrent sets that describe the system behavior on finest Morse sets. These sets depend upper semicontinuously on parameters. The connection with ergodic theory is provided through the construction of invariant measures based on chains.

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1 Introduction

This paper elaborates on some notions and results in the theory of dynamical systems in continuous time, due to C. Conley. We stress the relations between chain transitivity, Morse decompositions and attractors. While many of the individual results in this paper are known, they have not been presented in a unified way that explores all the connections. The paper is mostly self-contained (except for a few topological results), and presents several examples to stress the core concepts of global behavior. It is an extended version of Appendix B in [6].

In Section 2 we recall some basic properties of compact metric spaces. Section 3 discusses the basic concepts of continuous flows on compact metric spaces with time in the real line $\mathbb{R}$. Sections 4, 5, and 6 analyze the relations between Morse decompositions and attractors, Morse decompositions and chain recurrence, and chain recurrence and attractors, respectively. Section 7 is devoted to the construction of invariant measures based on chains. The final Section 8 considers families of dynamnical systems and shows that maximal chain transitive sets depend upper semicontinuously on parameters.

Conley's theory of flows on compact metric spaces also allows the construction of generalized Lyapunov functions outside of the chain recurrent set. For an elaboration of this point of view, see Robinson [21, Section 9.1] or Easton [10].

2 Metric Spaces

This paper considers continuous dynamical systems on compact metric spaces. For these, we recall a few basic concepts and theorems.

Definition 2.1 *A metric space (X, d) is a set X together with a distance function $d : X \times X \to \mathbb{R}$ such that for all points $x, y, z \in X$ the following holds: (i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$, (ii) $d(x, y) = d(y, x)$, and (iii) $d(x, z) \leq d(x, y) + d(y, z)$.*

A metric space X is compact if every sequence in X has a convergent subsequence. This is equivalent to each of the following conditions (cp., e.g., Pedersen [18, Theorem 1.6.2]):

(i) Each cover $X = \bigcup_\alpha V_\alpha$ by open subsets V_α with α in some index set has a finite subcover.

(ii) If Δ is a family of closed subsets of X, such that no intersection of finitely many sets in Δ is empty, then the intersection of all sets in Δ is nonempty.

Note that condition (ii) implies, in particular, that every decreasing (with respect to set inclusion) family of nonempty closed subsets of X has nonvoid intersection.

A metric space X is compact if and only if it is complete (i.e., each Cauchy sequence has a limit) and it is totally bounded; this property means that for every $\varepsilon > 0$ there are finitely many points $x_1, ..., x_n \in X$ such that

$$X = \bigcup_{i=1}^{n} \{y \in X,\ d(y, x_i) < \varepsilon\}$$

(see, e.g., Engelking [11, Theorem 4.3.29]). Furthermore, every compact metric space has a countable basis of its topology, i.e., there are countably many open sets V_n, $n \in \mathbb{N}$, such that every open set V can be written as the union of sets V_n.

Theorem 2.2 (Baire) *The countable intersection of open and dense subsets in a complete metric space is dense.*

A proof is given, e.g., in Pedersen [18, Proposition 2.2.2.] or Engelking [11, Corollary 3.9.4].

Theorem 2.3 (Blaschke) *The set of nonvoid closed subsets of a compact metric space becomes a compact metric space under the Hausdorff distance*

$$d_H(A, B) = \max \left\{ \max_{a \in A} \left[\min_{b \in B} d(a, b) \right], \max_{b \in B} \left[\min_{a \in A} d(a, b) \right] \right\}. \tag{1}$$

In fact, one can verify that this space is complete and totally bounded and hence compact.

3 Flows

We start with some basic concepts and properties for continuous dynamical systems on compact metric spaces with an emphasis on Conley's theory [7], [8]. For a thorough analysis see, in particular, Akin [1], Robinson [21], and Katok and Hasselblatt [14]. For generalizations to the case of noncompact metric spaces, see Rybakowski [23] and Hurley [13].

Definition 3.1 *A flow or continuous time dynamical system on a metric space X is given by a continuous map $\Phi : \mathbb{R} \times X \to X$ that satisfies $\Phi(0, x) = x$ and $\Phi(t + s, x) = \Phi(t, \Phi(s, x))$ for all $x \in X$ and all $t, s \in \mathbb{R}$.*

In the following we frequently use the suggestive notations $x \cdot t := \Phi_t x := \Phi(t, x)$ for $t \in \mathbb{R}$ and $x \in X$. The orbit of a point $x \in X$ is then $\{y \in X$, there is $t \in \mathbb{R}$ with $y = \Phi(t, x)\} = x \cdot \mathbb{R}$.

Definition 3.2 *The ω-limit set of a subset $Y \subset X$ is defined as*

$$\omega(Y) = \left\{ y \in X, \begin{array}{c} \text{there are } t_k \to \infty \text{ and } y_k \in Y \\ \text{such that } y_k \cdot t_k \to y \end{array} \right\} = \bigcap_{t>0} \mathrm{cl}\left(Y \cdot [t, \infty)\right).$$

Similarly

$$\omega^*(Y) = \left\{ y \in X, \begin{array}{c} \text{there are } t_k \to -\infty \text{ and} y_k \in Y \\ \text{such that } y_k \cdot t_k \to y \end{array} \right\} = \bigcap_{t>0} \mathrm{cl}\left(Y \cdot (-\infty, -t]\right).$$

Note that, in general, $\omega(Y)$ will be larger than the union of all $\omega(y)$, $y \in Y$, see Example below. If the space X is compact, the sets $\omega(Y)$ are nonvoid, compact, and invariant. They are connected if Y is connected. The ω^*-limit sets (often denoted as α-limit sets) are the ω-limit sets for the time reversed system $\Phi^*(t, x) := \Phi(-t, x)$, $t \in \mathbb{R}$, $x \in X$. A point $x \in X$ is called *recurrent* if $x \in \omega(x)$.

Example 3.3 *Consider the ordinary differential equation*

$$\dot{x} = x(x-1)(x-2)^2(x-3)$$

on the compact interval $X := [0, 3]$. The solutions $\varphi(t, x)$ of this equation with $\varphi(0, x) = x$ are unique and exist for all $t \in \mathbb{R}$. Hence they define a dynamical system $\Phi : \mathbb{R} \times [0, 3] \longrightarrow [0, 3]$ via $\Phi(t, x) := \varphi(t.x)$. The limit sets of this system are of the following form: For points $x \in [0, 3]$ we have

$$\omega(x) = \begin{cases} \{0\} & \text{for } x = 0 \\ \{1\} & \text{for } x \in (0, 2) \\ \{2\} & \text{for } x \in [2, 3) \\ \{3\} & \text{for } x = 3. \end{cases}$$

Limit sets for subsets of $[0, 3]$ can be entire intervals. E.g.,for $Y = [a, b]$ with $a \in (0, 1]$ and $b \in [2, 3)$ we have $\omega(Y) = [1, 2]$, which can be seen as follows: Obviously, it holds that $1, 2 \in \omega(Y)$. Let $x \in (1, 2)$, then $\lim_{t \to -\infty} \Phi(t, x) = 2$. We define $t_n := n \in \mathbb{N}$ and $x_n := \varphi(-n, x) \in (1, 2) \subset Y$. Then $\Phi(t_n, x_n) = \Phi(n, \Phi(-n, x)) = x$ for all $n \in \mathbb{N}$, which shows that $\omega(Y) \supset [1, 2]$. For the reverse inclusion let $x \in (0, 1)$. Note that $\lim_{t \to \infty} \Phi(t, a) = 1$ and for all $y \in [a, 1)$ and all $t \geq 0$ we have $d(\Phi(t, y), 1) \leq d(\Phi(t, a), 1)$, where $d(\cdot, \cdot)$ is the metric on $[0, 3]$ inherited from $\mathbb{R}$. Hence for any sequence y_n in $[a, 1)$ and any $t_n \to \infty$ one sees that $d(\Phi(t_n, y_n), 1) \leq d(\Phi(t_n, a), 1)$ and therefore $\lim_{n \to \infty} d(\Phi(t_n, y_n), 1) \leq \lim_{n \to \infty} d(\Phi(t_n, a), 1) = 0$. This implies that no point $x \in (0, 1]$ can be in $\omega(Y)$. The same argument applies to $x = 0$, and one argues similarly for $x \in (2, 3]$. Furthermore, the limit set of a subset Y can strictly include Y, e.g., for $Y = (0, 3)$ it holds that $\omega(Y) = [0, 3]$:
We show that $0, 3 \in \omega(Y)$, the rest follows easily. Let $x \in (0, 1)$. Define $y_n := \Phi(-2n, x)$ and $x_n := \Phi(-n, x)$, then $\Phi(n, y_n) = \Phi(n, \Phi(-2n, x)) = \Phi(-n, x) = x_n$ and $\lim x_n = 0$. Hence with $t_n := n$ and y_n as above we have $\Phi(t_n, y_n) \to 0$. The argument is similar for proving that $3 \in \omega(Y)$, and for points in $(0, 3)$.

Example 3.4 *Consider the following dynamical system Φ in $\mathbb{R}^2 \backslash \{0\}$, given by a differential equation in polar form for $r > 0$, $\theta \in [0, 2\pi)$, and $a \neq 0$:*

$$\dot{r} = 1 - r, \ \dot{\theta} = a.$$

For each $x \in \mathbb{R}^2 \backslash \{0\}$ the ω-limit set is the circle $\omega(x) = \mathbb{S}^1 = \{(r, \theta),\ r = 1, \theta \in [0, 2\pi)\}$.The state space $\mathbb{R}^2 \backslash \{0\}$ is not compact, and α-limit sets exist only for $y \in \mathbb{S}^1$, for which we have $\omega^*(y) = \mathbb{S}^1$.

Example 3.5 *For dynamical systems in $\mathbb{R}^2$ we have: A non-empty, compact limit set of a dynamical system in $\mathbb{R}^2$, which contains no fixed points, is a closed, i.e. a periodic orbit (theorem of Poincaré-Bendixson, see, e.g., [21]). Any non-empty, compact limit set of a dynamical system in $\mathbb{R}^2$ consists of fixed points, connecting orbits (such as homoclinic or heteroclinic orbits), and periodic orbits.*

Definition 3.6 *A flow on a metric space X is called topologically transitive if there exists some $x \in X$ such that $\omega(x) = X$; the flow is called topologically mixing if for any two open sets V_1, $V_2 \subset X$ there exists $T > 1$ such that*

$$V_1 \cdot (-T) \cap V_2 \neq \varnothing.$$

Proposition 3.7 *If a flow on a complete metric space is topologically mixing, it is topologically transitive and $\{x \in X,\ \omega(x) = X\}$ is residual, i.e., it contains a countable intersection of open and dense subsets.*

Proof. Topological mixing implies that for any two open sets V_1, $V_2 \subset X$ there exists a sequence $t_k \to \infty$ such that

$$V_1 \cdot (-t_k) \cap V_2 \neq \varnothing.$$

Thus for all open $V \subset X$ the set $\bigcup_{t \geq 0} V \cdot (-t)$ is dense in X, because otherwise there would exist open sets V_1 and V_2 with $\left(\bigcup_{t \geq 0} V_1 \cdot (-t)\right) \cap V_2 = \varnothing$. Now for a countable basis V_n of the topology and $m, n \in \mathbb{N}$ the sets $V_n \cdot (-m)$ are open. Then the sets

$$X_{m,n} := \bigcup_{t \geq m} V_n \cdot (-t) = \bigcup_{t \geq 0} (V_n \cdot (-m)) \cdot (-t)$$

are open and dense. Hence, by Baire's theorem (Theorem 2.2), the intersection $\bigcap_{m,n \in \mathbb{N}} X_{m,n}$ is nonvoid. We claim that for every x in this set $\omega(x) = X$. It suffices to show that the closure of every basis set V_n has nonvoid intersection with $\omega(x)$. Clearly, $x \in \bigcap_{m,n \in \mathbb{N}} X_{m,n} \subset \bigcap_{n \in \mathbb{N}} \bigcup_{t \geq m} (V_n \cdot (-t))$. This shows that $x \cdot t_m \in V_n$ for a sequence $t_m \to \infty$. ■

We note that related but different concepts of topological transitivity and topological mixing are, e.g., discussed in [14]. The next result due to Banks et al. [3] shows that a topologically transitive flow with a dense set of periodic points also has sensitive dependence on initial conditions. Thus it is chaotic in the sense of Devaney [9].

Definition 3.8 *A flow Φ on a metric space X has sensitive dependence on initial conditions if there is $\delta > 0$ such that for every $x \in X$ and every neighborhood N of x there are $y \in N$ and $T > 0$ such that $d(y \cdot T, x \cdot T) > \delta$.*

Proposition 3.9 *Consider a flow Φ on a metric space X that is not a single periodic orbit. If the flow is topologically transitive and has a dense subset of periodic points, then it has sensitive dependence on initial points.*

Proof. First observe that there is a number $\delta_0 > 0$ such that for all $x \in X$ there exists a periodic point $q \in X$ whose orbit is a distance at least $\delta_0/2$ from x. Indeed, choose two arbitrary periodic points q_1 and q_2 with disjoint orbits $q_1 \cdot \mathbb{R}$ and $q_2 \cdot \mathbb{R}$. Let δ_0 denote the distance between the compact sets $q_1 \cdot \mathbb{R}$ and $q_2 \cdot \mathbb{R}$. Then by the triangle inequality, every point $x \in X$ is a distance at least $\delta_0/2$ from one of the chosen two periodic orbits. We will show that Φ has sensitive dependence on initial conditions with sensitivity constant $\delta = \delta_0/8$.

Let x be an arbitrary point in X and let N be some neighborhood of x. Because the periodic points of Φ are dense, there exists a periodic point p in the intersection $U = N \cap B_\delta(x)$ of N with the open ball $B_\delta(x)$ of radius δ centered at x. Let T denote the period of p. As we showed earlier, there exists a periodic point $q \in X$ whose orbit is a distance at least 4δ from x. Set

$$V = \bigcap_{0 \le t \le T} \left(B_\delta(q \cdot t) \cdot (-t) \right).$$

By continuous dependence on the initial value, the set V is open and nonvoid because $q \in V$. Consequently, because Φ is topologically transitive, there exist y in U and $\tau > 0$ such that $y \cdot \tau \in V$.

Now let j be the integer part of $\tau/T + 1$. Then $0 \le jT - \tau \le T$ and, by construction, one has

$$y \cdot (jT) = (y \cdot \tau) \cdot (jT - \tau) \in V \cdot (jT - \tau) \subset B_\delta(q \cdot (jT - \tau)).$$

Now $p \cdot (jT) = p$, and so by the triangle inequality

$$\begin{aligned} d(p \cdot (jT), y \cdot (jT)) &= d(p, y \cdot (jT)) \\ &\ge d(x, q \cdot (jT - \tau)) - d(q \cdot (jT - \tau), y \cdot (jT)) - d(p, x). \end{aligned}$$

Consequently, because $y \cdot (jT) \in B_\delta(q \cdot (jT - \tau))$, one has

$$d(p \cdot (jT), y \cdot (jT)) > 4\delta - \delta - \delta = 2\delta.$$

Thus, using the triangle inequality again, either $d(x \cdot (jT), y \cdot (jT)) > \delta$ or $d(x \cdot (jT), p \cdot (jT)) > \delta$. In either case, we have found a point in N whose image after time jT is more than distance δ from the image of x. ■

Remark 3.10 *The proof of Proposition 3.9 is adapted from discrete to continuous time from Banks et al. [3].*

4 Morse Decompositions and Attractors

The global behavior of flows on compact metric spaces can be described via Morse decompositions, which are special collections of compact invariant subsets. A set $K \subset X$ is called *invariant* if $x \cdot \mathbb{R} \subset K$ for all $x \in K$; a compact subset $K \subset X$ is called *isolated invariant*, if it is invariant and there exists a neighborhood N of K, i.e., a set N with $K \subset \operatorname{int} N$, such that $x \cdot \mathbb{R} \subset N$ implies $x \in K$. Thus an invariant set K is isolated if every trajectory that remains close to K actually belongs to K.

Example 4.1 *Consider the dynamical system discussed in Example 3.3. Invariant sets for this system are the sets of the form* $\{x^*\}$*, where* x^* *is a fixed point, all closed intervals with fixed points at the boundaries, and disjoint unions of these two types. Note that 'invariant' means forward (for* $t \geq 0$*) and backward (for* $t \leq 0$*) in time, hence this flow has no other invariant sets. It is easily proved that all invariant sets of this system are isolated invariant.*

Example 4.2 *Consider on the interval* $[0,1] \subset \mathbb{R}$ *the ordinary differential equation*

$$\dot{x} = \begin{cases} x^2 \sin(\frac{\pi}{x}) & \text{for} \quad x \in (0,1] \\ 0 & \text{for} \quad x = 0. \end{cases}$$

Invariant sets for the associated flow include again sets of the form $\{x^*\}$ *where* x^* *is a fixed point. But the set* $\{0\}$ *is not isolated invariant: Let* $U(0,\varepsilon)$ *be the* $\varepsilon-$*neighborhood of* 0 *in* $[0,1]$*. Then there exists* $x \in U(0,\varepsilon)$ *with* $\sin(\frac{\pi}{x}) = 0$*, i.e.* x *is a fixed pint and hence* $\Phi(t,x) = x \in U(0,\varepsilon)$ *for all* $t \in \mathbb{R}$*.*

Definition 4.3 *A Morse decomposition of a flow on a compact metric space is a finite collection* $\{\mathcal{M}_i,\ i = 1,...,n\}$ *of nonvoid, pairwise disjoint, and isolated compact invariant sets such that:(i) For all* $x \in X$ *one has* $\omega(x),\ \omega^*(x) \subset \bigcup_{i=1}^{n} \mathcal{M}_i$*. (ii) Suppose there are* $\mathcal{M}_{j_0}, \mathcal{M}_{j_1}, ..., \mathcal{M}_{j_l}$ *and* $x_1, ..., x_l \in X \setminus \bigcup_{i=1}^{n} \mathcal{M}_i$ *with* $\omega^*(x_i) \subset \mathcal{M}_{j_{i-1}}$ *and* $\omega(x_i) \subset \mathcal{M}_{j_i}$ *for* $i = 1,...,l$*; then* $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_l}$*. The elements of a Morse decomposition are called Morse sets.*

Thus the Morse sets contain all limit sets and "cycles" are not allowed. As an easy consequence of this definition we obtain the following equivalent characterization.

Proposition 4.4 *A finite collection* $\{\mathcal{M}_i,\ i = 1,...,n\}$ *of nonvoid, pairwise disjoint, and isolated compact invariant sets is a Morse decomposition if and only if condition (i) holds,* $\omega^*(x) \cup \omega(x) \subset \mathcal{M}_i$ *implies* $x \in \mathcal{M}_i$*, and the following relation "*$\preceq$*" is an order (satisfying reflexivity, transitivity and antisymmetry):*

$$\mathcal{M}_i \preceq \mathcal{M}_k \text{ if } \begin{array}{c} \text{there are } \mathcal{M}_{j_0} = \mathcal{M}_i, \mathcal{M}_{j_1}, ...\mathcal{M}_{j_l} = \mathcal{M}_k \text{ and } x_1, ..., x_l \in X \\ \text{with } \omega^*(x_k) \subset \mathcal{M}_{j_{k-1}} \text{ and } \omega(x_k) \subset \mathcal{M}_{j_k} \text{ for } k = 1, ..., l. \end{array} \tag{2}$$

We enumerate the Morse sets in such a way that $\mathcal{M}_i \preceq \mathcal{M}_j$ *implies* $i \leq j$*.*

Proof. The "no-cycle" condition (ii) in the definition of Morse decompositions is equivalent to the stated property of the limit sets and the antisymmetry property of the order "$\preceq$". Transitivity is clear and reflexivity follows from invariance of the Morse sets. The numbering is always possible, but it need not be unique. ■

Note that $i < j$ does not imply $\mathcal{M}_i \preceq \mathcal{M}_j$ and that it does not imply the existence of $x \in X$ with $\omega^*(x) \subset \mathcal{M}_i$ and $\omega(x) \subset \mathcal{M}_j$. Morse decompositions describe the flow via its movement from Morse sets with lower indices toward those with higher ones.

A Morse decomposition $\{\mathcal{M}_1, ..., \mathcal{M}_n\}$ is called *finer* than a Morse decomposition $\{\mathcal{M}'_1, ..., \mathcal{M}'_{n'}\}$, if for all $j \in \{1, ..., n'\}$ there is $i \in \{1, ..., n\}$ with $\mathcal{M}_i \subset \mathcal{M}'_j$. The intersection of two Morse decompositions $\{\mathcal{M}_1, ..., \mathcal{M}_n\}$ and $\{\mathcal{M}'_1, ..., \mathcal{M}'_{n'}\}$ defines a Morse decomposition

$$\left\{\mathcal{M}_i \cap \mathcal{M}'_j\,,\ i,\, j\right\},$$

where only those indices $i = 1, ..., n$, $j = 1, ..., n'$ with $\mathcal{M}_i \cap \mathcal{M}'_j \neq \varnothing$ are allowed. Note that, in general, intersections of infinitely many Morse decompositions do not define a Morse decomposition. In particular, there need not exist a finest Morse decomposition. The intersection of all Morse decompositions for a flow need not be a countable set. It may form a Cantor set; see [1, p.25] (and use Theorems 4.16 and 6.4).

Example 4.5 *Consider the dynamical system discussed in Example 3.3. This flow has, e.g., the following Morse decompositions* $[1,3] \preceq \{0\}$, $\{0\} \succeq \{1\} \preceq [2,3]$, $\{0\} \succeq [1,2] \preceq \{3\}$, $\{1\} \preceq \{0\} \cup [2,3]$, *and others. It also has a unique finest Morse decomposition* $\{0\} \succeq \{1\} \preceq \{2\} \preceq \{3\}$.

Example 4.6 *Consider the dynamical system defined in Example 4.2. Morse decompositions of the associated flow are, e.g., the sets* $\mathcal{M}^n := \{\{\frac{1}{n}\}, [0, \frac{1}{n+1}] \cup [\frac{1}{n-1}, 1]\}$ *for* $n \in \mathbb{N}$. *Note that* $\cap \mathcal{M}^n = \{\{0\}, \{\frac{1}{n}\}$ *for* $n \in \mathbb{N}\}$ *is not a Morse decomposition. This system does not have a finest Morse decomposition, since all the individual sets* $\{\frac{1}{n}\}$ *for* $n \in \mathbb{N}$ *would have to be included as Morse sets.*

Morse decompositions can be constructed from attractors and their complementary repellers. We will now define these rather intricate objects.

Definition 4.7 *For a flow on a compact metric space* X *a compact invariant set* A *is an attractor if it admits a neighborhood* N *such that* $\omega(N) = A$. *A repeller is a compact invariant set* R *that has a neighborhood* N^* *with* $\omega^*(N^*) = R$.

We also allow the empty set as an attractor. A neighborhood N as in Definition 4.7 is called an *attractor neighborhood*. Every attractor is compact and invariant, and a repeller is an attractor for the time reversed flow. Furthermore, if A is an attractor in X and $Y \subset X$ is a compact invariant set, then $A \cap Y$ is an attractor for the flow restricted to Y.

Example 4.8 *Consider again the dynamical system discussed in Example 3.3. This system has, besides* $\varnothing$ *and the entire space* $[0,3]$, *three attractors, namely* $\{1\}$, $[1,2]$, *and* $[1,3]$. *The fact that these sets are indeed attractors follows directly from the limit sets discussed in Example 3.3. To see that there are no other attractors one argues as in Examples 3.3 and 4.1. Similarly, the nontrivial repellers of this system are seen to be* $\{0\}$, $[2,3]$, $\{3\}$, $\{0\} \cup [2,3]$, *and* $\{0\} \cup \{3\}$.

Example 4.9 *Consider the complete metric space* $\mathbb{S}^1$*, the* $1-$*dimensional sphere, which we identify here with* $\mathbb{R}/2\pi$*. On* $\mathbb{S}^1$ *the differential equation*

$$\dot{x} = \sin^2 x$$

defines a dynamical system. For this flow, the only attractors are $\varnothing$ *and* $\mathbb{S}^1$*: Let* $A \subset \mathbb{S}^1$ *be an attractor, i.e. there exists a neighborhood* $N(A)$ *with* $\omega(N) = A$*. For each point* $x \in \mathbb{S}^1$ *the limit set* $\omega(x)$ *contains at least one of the two fixed points* 0 *or* π*, which implies that each attractor has to contain at least one of the fixed points. Consider the point* π *and let* $N(\pi)$ *be any neighborhood. We have* $[\pi, 0] \subset \omega(N) \subset A$*. Repeating this argument for the fixed point* 0*, we see that* $[0, \pi] \subset A$*, and hence* $A = \mathbb{S}^1$*.*

We note the following lemma.

Lemma 4.10 *For every attractor neighborhood* N *of an attractor* A *there is a time* $t^* > 0$ *with* $\mathrm{cl}\,(N \cdot [t^*, \infty)) \subset \mathrm{int}\, N$*.*

Proof. We may assume that N is closed. Suppose that there are $t_n \to \infty$ and $x_n \in N$ with $x_n \cdot t_n \notin \mathrm{int}\, N$. Hence we may assume that $x_n \cdot t_n$ converges to some element $x \notin \mathrm{int}\, N$. This contradicts the assumption $\omega(N) = A$. ■

Lemma 4.11 *For an attractor* A*, the set* $A^* = \{x \in X,\ \omega(x) \cap A = \varnothing\}$ *is a repeller, called the complementary repeller. Then* (A, A^*) *is called an attractor-repeller pair.*

Proof. Let N be a compact attractor neighborhood of A. Choose $t^* > 0$ such that $\mathrm{cl}\,(N \cdot [t^*, \infty)) \subset N$ and define an open set V by

$$V = X \setminus \mathrm{cl}\,(N \cdot [t^*, \infty)).$$

Then $X = N \cup V$. Furthermore $V \cdot (-\infty, -t^*] \subset X \setminus N$ and therefore V is a neighborhood of $\omega^*(V) \subset X \setminus N \subset V$. Hence $\omega^*(V)$ is a repeller. Furthermore, by invariance $\omega^*(V) \subset A^*$. The converse inclusion follows, because A is isolated invariant. ■

Note that A and A^* are disjoint. There is always the trivial attractor-repeller pair $A = X,\ A^* = \varnothing$.

Example 4.12 *Consider again the dynamical system discussed in Examples 3.3 and 4.8. The nontrivial attractor-repeller pairs of this system are* $A_1 = \{1\}$ *with* $A_1^* = \{0\} \cup [2, 3]$*,* $A_2 = [1, 2]$ *with* $A_2^* = \{0\} \cup \{3\}$*, and* $A_3 = [1, 3]$ *with* $A_3^* = \{0\}$*.*

A consequence of the following proposition is, in particular, that in the time reversed system the complementary repeller of A^* is A.

Proposition 4.13 *If* (A, A^*) *is an attractor-repeller pair and* $x \notin A \cup A^*$*, then* $\omega^*(x) \subset A^*$ *and* $\omega(x) \subset A$*.*

Proof. By definition of A^* it follows that $\omega(x) \cap A \neq \varnothing$. Thus there is $t_0 > 0$ with $x \cdot t_0 \in N$, where N is a neighborhood of the attractor A with $\omega(N) = A$. Hence there cannot exist a point $y \in \omega(x) \backslash A$, and hence $\omega(x) \subset A$. Now suppose that there is $y \in \omega^*(x) \backslash A^*$. Thus by definition of A^* one has $\omega(y) \cap A \neq \varnothing$. Using continuous dependence on the initial value one finds that there are $t_n \to \infty$ with $x \cdot (-t_n) \to A$, and thus for n large enough, $x \cdot (-t_n) \in N$. Clearly $x \cdot (-t_n) \cdot t_n \to x$ and hence $\omega(N) = A$ implies that $x \in A$, contradicting the choice of x. Thus $\omega^*(x) \subset A^*$. ■

Trajectories starting in a neighborhood of an attractor leave the neighborhood in backwards time.

Lemma 4.14 *For a flow on a compact metric space X a compact invariant set A is an attractor if and only if there exists a compact neighborhood N of A such that $x \cdot (-\infty, 0] \not\subseteq N$ for all $x \in N \setminus A$.*

Proof. The necessity of the condition is clear because $x \cdot (-\infty, 0] \subset N$ implies $x \in \omega(N)$. Conversely, let N be a compact neighborhood of A such that $x \cdot (-\infty, 0] \not\subseteq N$ for all $x \in N \setminus A$. Thus there exists a $t^* > 0$ such that $x \cdot [-t^*, 0] \not\subseteq N$ for all $x \in N \cap \mathrm{cl}(X \setminus N)$. Now choose a neighborhood V of A such that $V \cdot [0, t^*] \subset N$. Then $V \cdot [0, \infty) \subset N$ and hence $\omega(V) = A$ and A is an attractor. ■

This implies the following characterization of attractor-repeller pairs.

Lemma 4.15 *Let $(x, t) \mapsto x \cdot t$ be a flow on a compact metric space X. Then a pair A, A^* of disjoint compact invariant sets is an attractor-repeller pair if and only if (i) $x \in X \setminus A^*$ implies $x \cdot [0, \infty) \cap N \neq \varnothing$ for every neighborhood N of A, and (ii) $x \in X \setminus A$ implies $x \cdot (-\infty, 0] \cap N^* \neq \varnothing$ for every neighborhood N^* of A^*.*

Proof. Certainly, these conditions are necessary. Conversely, suppose that (i) holds and let W be a compact neighborhood of A with $W \cap A^* = \varnothing$. Then (ii) implies that $x \cdot (-\infty, 0] \not\subseteq W$ for all $x \in W \setminus A$. By Lemma 4.14 this implies that A is an attractor. Moreover, it follows from (i) that $\omega(x) \cap A \neq \varnothing$ for all $x \in X \setminus A^*$. Hence $A^* = \{x \in X,\ \omega(x) \cap A = \varnothing\}$ is the complementary repeller of A. ■

The following result characterizes Morse decompositions via attractor-repeller sequences (it is often taken as a definition; cp. Rybakowski [23, Definition III.1.5 and Theorem III.1.8], Salamon [24], or Salamon and Zehnder [25].

Theorem 4.16 *For a flow on a compact metric space X a finite collection of subsets $\{\mathcal{M}_1, ..., \mathcal{M}_n\}$ defines a Morse decomposition if and only if there is a strictly increasing sequence of attractors*

$$\varnothing = A_0 \subset A_1 \subset A_2 \subset ... \subset A_n = X,$$

such that

$$\mathcal{M}_{n-i} = A_{i+1} \cap A_i^* \text{ for } 0 \leq i \leq n-1.$$

Proof. (i) Suppose that $\{\mathcal{M}_1, ..., \mathcal{M}_n\}$ is a Morse decomposition. Define a strictly increasing sequence of invariant sets by $A_0 = \varnothing$ and

$$A_k = \{x \in X,\ \omega^*(x) \subset \mathcal{M}_n \cup ... \cup \mathcal{M}_{n-k+1}\} \text{ for } k = 1, ..., n.$$

First we show that the sets A_k are closed. Clearly, the set $A_n = X$ is closed. Proceeding by induction, assume that A_{k+1} is closed and consider $x_i \in A_k$ with $x_i \to x$. We have to show that $\omega^*(x) \subset \mathcal{M}_n \cup ... \cup \mathcal{M}_{n-k+1}$. The induction hypothesis implies that $x \in A_{k+1}$ and hence we have $\omega^*(x) \subset \mathcal{M}_n \cup ... \cup \mathcal{M}_{n-k}$. Because $\omega^*(x) \subset \mathcal{M}_j$ for some $j \in \{1, ..., n\}$, either the assertion holds or $\omega^*(x) \subset \mathcal{M}_{n-k}$. In order to see that the latter case cannot occur, let V be an open neighborhood of $\mathcal{M}_{n-k}$ such that $V \cap \mathcal{M}_j = \varnothing$ for $j \neq n-k$. There are a sequence $t_\nu \to \infty$ and $z \in \mathcal{M}_{n-k}$ such that $x \cdot (-t_\nu) \in V$ and $d(x \cdot (-t_\nu), z) \leq \nu^{-1}$ for all $\nu \geq 1$. Hence for every ν there is a $m_\nu \geq \nu$ such that $x_{m_\nu} \cdot (-t_\nu) \in V$ and $d(x_{m_\nu} \cdot (-t_\nu), z) \leq 2\nu^{-1}$. Because $\omega(x_i) \cup \omega^*(x_i) \subset \mathcal{M}_n \cup ... \cup \mathcal{M}_{n-k+1}$ for all i, there are $\tau_\nu < t_\nu < \sigma_\nu$ such that $x_{m_\nu} \cdot (-\sigma_\nu)$ and $x_{m_\nu} \cdot (-\tau_v) \in \partial V$ and $x_{m_\nu} \cdot (-t) \in \operatorname{cl} V$ for all $t \in [\tau_\nu, \sigma_\nu]$. Invariance of $\mathcal{M}_{n-k}$ implies that $t_\nu - \tau_\nu \to \infty$ as $\nu \to \infty$. We may assume that there is $y \in \partial V$ with $x_{m_\nu} \cdot (-\sigma_\nu) \to y$ for $\nu \to \infty$. Then it follows that $y \cdot [0, \infty) \subset \operatorname{cl} V$ and hence by the choice of V one has $\omega(y) \subset \mathcal{M}_{n-k}$. Because A_{k+1} is closed and invariant, we have $y \in A_{k+1}$ and so $\omega^*(y) \subset \mathcal{M}_n \cup ... \cup \mathcal{M}_{k-n}$. The ordering of the Morse sets implies that $y \in \mathcal{M}_{n-k}$, contradicting $y \in \partial V$.

If A_k is not an attractor, Lemma 4.14 implies that for every neighborhood N of A_k there is $x \in N \setminus A_k$ with $x \cdot (-\infty, 0] \subset N$. Then there is $j \geq n-k+1$ with $\omega^*(x) \subset \mathcal{M}_j$. On the other hand $x \notin A_k$ implies $\omega^*(x) \not\subset \mathcal{M}_n \cup ... \cup \mathcal{M}_{n-k+1}$, hence $\omega^*(x) \in \mathcal{M}_i$ for some $i < n-k+1$. This contradiction implies that A_k is an attractor.

It remains to show that $\mathcal{M}_{n-i} = A_{i+1} \cap A_i^*$. Clearly, $\mathcal{M}_{n-i} \subset A_{i+1}$. Suppose that $x \in \mathcal{M}_{n-i} \setminus A_i^*$. Then $\omega(x) \subset A_i$ and therefore $\omega(x) \subset \mathcal{M}_j$ for some $j \geq n-i+1$. This contradiction proves $\mathcal{M}_{n-i} \subset A_{i+1} \cap A_i^*$. If conversely, $x \in A_{i+1} \cap A_i^*$, then $\omega^*(x) \subset \mathcal{M}_n \cup ... \cup \mathcal{M}_{n-i}$. From $x \in A_i^*$ we conclude

$$\omega(x) \cap \mathcal{M}_n \cup ... \cup \mathcal{M}_{n-i+1} \subset \omega(x) \cap A_i = \varnothing$$

and hence $\omega(x) \subset \mathcal{M}_1 \cup ... \cup \mathcal{M}_{n-i}$. Now the definition of a Morse decomposition implies $x \in \mathcal{M}_{n-i}$.

(ii) Conversely, let the sets $\mathcal{M}_j$, $i = 1, ..., n$, be defined by an increasing sequence of attractors as indicated earlier. Clearly these sets are compact and invariant. If $i < j$, then $\mathcal{M}_{n-i} \cap \mathcal{M}_{n-j} = A_{i+1} \cap A_i^* \cap A_{j+1} \cap A_j^* = A_{i+1} \cap A_j^* \subset A_j \cap A_j^* = \varnothing$; hence the sets $\mathcal{M}_i$ are pairwise disjoint. It remains to prove that for $x \in X$ either $x \cdot \mathbb{R} \subset \mathcal{M}_j$ for some j or else there are indices $i < j$ such that $\omega^*(x) \subset \mathcal{M}_{n-j}$ and $\omega(x) \subset \mathcal{M}_{n-i}$. There is a smallest integer i such that $\omega(x) \subset A_i$, and there is a largest integer j such that $\omega^*(x) \subset A_j^*$. Clearly $i > 0$ and $j < n$. Now $\omega(x) \not\subset A_{i-1}$, i.e., $x \in A_{i-1}^*$. Thus by invariance $x \cdot \mathbb{R} \subset A_{i-1}^*$ and $\omega(x) \subset A_{i-1}^*$. On the other hand, $\omega^*(x) \not\subset A_{j+1}^*$ and we claim that $x \cdot \mathbb{R} \subset A_{j+1}$. In fact, otherwise $x \cdot t \notin A_{j+1}$ for some $t \in \mathbb{R}$. If now $x \cdot t \notin A^*_{j+1}$, then $\omega^*(x) \subset A^*_{j+1}$, a contradiction, thus $x \cdot t \in A^*_{j+1}$ and so $\omega(x) \subset A^*_{j+1}$, again a contradiction. Hence indeed $x \cdot \mathbb{R} \subset A_{j+1}$.

Now $j \geq i-1$, because otherwise $j+1 \leq i-1$ and thus $A_{j+1} \subset A_{i-1}$, which implies $x \cdot \mathbb{R} \subset A^*_{i-1} \cap A_{i-1} = \varnothing$. If $j = i-1$, then $x \cdot \mathbb{R} \subset A^*_{i-1} \cap A_i = \mathcal{M}_{n-i-1}$. If $j > i-1$, then $\omega(x) \subset A^*_{i-1} \cap A_{j+1} \subset A^*_{i-1} \cap A_i = \mathcal{M}_{n-i+1}$ and $\omega^*(x) \subset A^*_j \cap A_{j+1} = \mathcal{M}_{n-j}$. This proves the claim. ■

Corollary 4.17 *Let $\{M_i,\ i = 1, ..., n\}$ be the finest Morse decomposition of a flow on a compact metric space, with order $\preceq$. Then the maximal (with respect to $\preceq$) Morse sets are attractors, and the minimal Morse sets are repellers.*

Proof. The results follows direcftly from Proposition 4.13 and Lemma 4.14. ■

Example 4.18 *We illustrate Theorem 4.16 by looking again at Example 3.3. For this system a strictly increasing sequence of attractors with their corresponding repellers is*

$$A_0 = \emptyset \subset A_1 = \{1\} \subset A_2 = [1,2] \subset A_3 = [1,3] \subset A_4 = [0,3],$$
$$A^*_0 = [0,3] \supset A^*_1 = \{0\} \cup [2,3] \supset A^*_2 = \{0\} \cup \{3\} \supset A^*_3 = \{0\} \supset A^*_4 = \varnothing.$$

The associated Morse decomposition is

$$\mathcal{M}_4 = A_1 \cap A^*_0 = \{1\},\ \mathcal{M}_3 = A_2 \cap A^*_1 = \{2\},$$
$$\mathcal{M}_2 = A_3 \cap A^*_2 = \{3\},\ \mathcal{M}_1 = A_4 \cap A^*_3 = \{0\}.$$

Example 4.19 *Consider the dynamical system defined in Example 4.9. According to Theorem 4.16 its only Morse decomposition is the trivial one $\mathcal{M} = \{\mathbb{S}^1\}$. This can also be seen directly from Definition 4.3 of a Morse decomposition: The union of the Morse sets needs to contain all limit sets of individual points $x \in \mathbb{S}^1$. In this example we have*

$$\omega(x) = \begin{cases} \{\pi\} & \text{for } x \in (0,\pi] \\ \{0\} & \text{for } x \in (\pi, 0]. \end{cases}$$

and

$$\omega^*(x) = \begin{cases} \{0\} & \text{for } x \in [0,\pi) \\ \{\pi\} & \text{for } x \in [\pi, 0). \end{cases}$$

Assume that there are two Morse sets $\mathcal{M}_1$ and $\mathcal{M}_2$, with $0 \in \mathcal{M}_1$ and $\pi \in \mathcal{M}_2$. This violates the no-cycle condition (ii) of Definition 4.3. Hence the points 0 and π are in the same Morse set and the only Morse set is $\mathcal{M} = \mathbb{S}^1$.

5 Morse Decompositions and Chain Recurrence

We will now introduce the concept of chain recurrence and elaborate its relation to Morse decompositions.

Definition 5.1 *For x, $y \in X$ and $\varepsilon, T > 0$ an (ε, T)-chain from x to y is given by a natural number $n \in \mathbb{N}$, together with points*

$$x_0 = x, x_1, ..., x_n = y \in X \text{ and times } T_0, ...T_{n-1} \geq T,$$

such that $d(x_i \cdot T_i, x_{i+1}) < \varepsilon$ for $i = 0, 1, ..., n-1$.

Note that the number n of "jumps" is not bounded. Hence one may introduce "trivial jumps." Furthermore, as the notation suggests, only small values of $\varepsilon > 0$ are of interest.

Definition 5.2 *A subset $Y \subset X$ is chain transitive if for all $x, y \in Y$ and all $\varepsilon, T > 0$ there exists a (ε, T)-chain from x to y. A point $x \in X$ is chain recurrent if for all $\varepsilon, T > 0$ there exists a (ε, T)-chain from x to x. The chain recurrent set $\mathcal{R}$ is the set of all chain recurrent points.*

Note that we do not require in this definition that the considered (ε, T)-chains lie in Y. It is easily seen that $\mathcal{R}$ is closed and invariant.

Example 5.3 *Consider again the dynamical system discussed in Example 3.3. Obviously, all fixed points are chain recurrent points: just pick any $t_n > 0$ and any $\varepsilon > 0$ and consider chains of the type $x_n = \Phi(t_n, x_{n-1})$ with $x_0 = x$. In this example, there are no other points with this property, which can be seen as follows: Consider a point $x \in [0, 3]$ that is not a fixed point and let $\delta := \min d(x, x^*)$, where x^* is a fixed point. Let $\varepsilon := \frac{1}{3}\delta$ and $a := \lim_{t\to\infty} \Phi(t, x)$. Let $T := \min\{t > 0,\ d(\Phi(t, x), a) = \varepsilon\}$. Fix ε, T and consider $(\varepsilon, T)-$chains starting in x: $x_0 = x$, $y_1 = \Phi(t, x)$ for some $t \geq T$, then $d(y_1, a) \leq \varepsilon$, since convergence of $\Phi(t, x)$ to a is monotone. Pick $x_1 \in U(y_1, \varepsilon)$ the $\varepsilon-$neighborhood of y_1. Then $d(x, x_1) > \varepsilon$ and there are two possibilities: (a) $x_1 \notin \{\Phi(t, x),\ t \geq 0\}$, in this case $d(\{\Phi(t, x_1),\ t \geq 0\}, x) \geq 3\varepsilon$. (b) $x_1 \in \{\Phi(t, x),\ t \geq 0\}$, in this case $d(\Phi(t, x_1), a)) \leq \varepsilon$ for all $t \geq T$. Repeating the construction for $y_2 := \Phi(t, x_1)$ for some $t \geq T$ and $x_2 \in U(y_2, \varepsilon)$ we see that for all $n \in \mathbb{N}$ it holds that $d(x_n, x) > \varepsilon$, and hence there is no $(\varepsilon, T)-$chain from x to x. The key to this example is that trajectories starting from x 'move away' and cannot return, even using jumps of size ε, to x or $\omega^*(x)$, because of the topology of the state space $[0, 3]$. This is different in the following example.*

Example 5.4 *Consider the dynamical system defined in Example 4.9. In this case we have $\mathcal{R} = \mathbb{S}^1$: Let $x \in \mathbb{S}^1$ and $\varepsilon, T > 0$ be given, assume without loss of generality that $x \in (0, \pi]$. Since $\lim_{t\to\infty} \Phi(t, x) = \pi$ there is $t_1 > T$ with $d(\Phi(t_1, x), \pi) < \frac{\varepsilon}{2}$. Pick $x_1 \in U(\pi, \frac{\varepsilon}{2}) \cap (\pi, 0)$. Because of $\lim_{t\to\infty} \Phi(t, x_1) = 0$ there is $t_2 > T$ with $d(\Phi(t_2, x), 0) < \frac{\varepsilon}{2}$. Furthermore $\lim_{t\to-\infty} \Phi(t, x) = 0$ and hence there is $t_3 > T$ with $x_2 := \Phi(-t_3, x) \in U(0, \frac{\varepsilon}{2})$. Now $x = x_0$, x_1, x_2, $x_3 = x$ is an $(\varepsilon, T)-$ chain from x to x.*
In a similar way one constructs for any $\varepsilon, T > 0$ an $(\varepsilon, T)-$ chain from x to y for any two points $x, y \in \mathbb{S}^1$, showing that this dynamical system is chain transitive, and hence chain recurrent on $\mathbb{S}^1$.

The next proposition shows that in $\mathcal{R}$ only the existence of a positive lower bound for the times in (ε, T)-chains is important.

Proposition 5.5 *Consider $y \in \mathcal{R}$ and $x \in X$ and let $\tau > 0$. If for every $\varepsilon > 0$ there exists an (ε, τ)-chain from x to y, then for every $\varepsilon, T > 0$ there exists an (ε, T)-chain from x to y.*

Proof. It suffices to show that for every $\varepsilon > 0$ there is an $(\varepsilon, 2\tau)$-chain from x to y. By compactness of X the map Φ is uniformly continuous on $X \times [0, 3\tau]$. Hence there is $\delta \in \left(0, \frac{\varepsilon}{2}\right)$ such that for all $a, b \in X$ and $t \in [0, 3\tau]$:

$$d(a, b) < \delta \text{ implies } d(a \cdot t, b \cdot t) < \frac{\varepsilon}{2}.$$

Now let a (δ, τ)-chain $x_0 = x,\ x_1, ..., x_m = y$ with times $\tau_0, ..., \tau_{m-1} \geq \tau$ be given. We may assume that $\tau_i \in [\tau, 2\tau]$. We may assume that $m \geq 2$, because we may concatenate this chain with a chain from y to y. Thus there are $q \in \{0, 1, ...\}$ and $r \in \{2, 3\}$ with $m = 2q + r$. We obtain an $(\varepsilon, 2\tau)$-chain from x to y given by points

$$y_0 = x,\ y_1 = x_2,\ y_2 = x_4, ...,\ y_q = x_{2q},\ y_{q+1} = x_m = y$$

with times

$$T_0 = \sum_{i=0}^{1} \tau_i,\ T_1 = \sum_{i=2}^{3} \tau_i, ...,\ T_q = \sum_{i=2q}^{m} \tau_i.$$

This follows by the triangle inequality and the choice of δ. ■

Theorem 5.6 *The flow restricted to a maximal (with respect to set inclusion) chain transitive subset of the chain recurrent set $\mathcal{R}$ is chain transitive. In particular, the flow restricted to $\mathcal{R}$ is chain recurrent.*

Proof. Let $y, y' \in Y \subset \mathcal{R}$, where Y is a maximal chain transitive set in $\mathcal{R}$. For every $p \in \mathbb{N}$ there is an $(1/p, 1)$-chain in X from y to y', say with $x_0 = y,\ x_1, ..., x_{n_p} = y' \in X$ and times $T_0^p, ..., T_{n_p-1}^p \in [1, 2]$. Define $K_p = \bigcup_{i=0}^{n_p} \{x_i \cdot [0, T_i^p]\}$. By Blaschke's theorem (Theorem 2.3), there exists a subsequence of K_p converging in the Hausdorff metric d_H to some nonvoid compact subset $K \subset X$ with $y, y' \in K$. We claim that for all $x,\ z \in K$ and all $q \in \mathbb{N}$ there is an $(1/q, 1)$-chain in K with times $T_0^q, ..., T_{n_q-1}^q \in [1, 2]$ from x to z. In particular, this implies $K \subset Y$ and hence the assertion follows.

The claim is proved as follows. The flow is uniformly continuous on the compact set $X \times [0, 2]$. Hence there is a number $\delta > 0$ such that

$$d(a, b) < \delta \text{ implies } d(a \cdot t, b \cdot t) < \frac{1}{3q} \text{ for all } t \in [0, 3].$$

Choosing $p \in \mathbb{N}$ with $p > \max\left\{3q, \delta^{-1}\right\}$and $d_H(K_p, K) < \delta$ one can construct a $(\frac{1}{q}, 1)$-chain from x to z in K as required. ■

Proposition 5.7 *A closed subset Y of a compact metric space X is chain transitive if it is chain recurrent and connected. Conversely, if the flow on X is chain transitive, then X is connected.*

Proof. Suppose first that Y is chain recurrent and connected. Let $x, y \in Y$ and fix $\varepsilon, T > 0$. Cover Y by balls of radius $\varepsilon/4$. By compactness there are finitely many points, say $y_1, ..., y_{n-1} \in Y$ such that for all $z \in Y$ there is y_i with $d(z, y_i) < \varepsilon/4$. Define $y_0 = x$ and $y_n = y$. Because Y is connected the distance between the points y_i is bounded below by $\frac{3}{4}\varepsilon$. Now use that by chain recurrence of the flow there are $(\varepsilon/4, T)$-chains from y_i to y_i for $i = 0, 1, ...n-1$. Appropriate concatenation of these chains leads to an (ε, T)-chain from x to y. Hence chain transitivity follows.

Conversely, let the flow on X be chain transitive. If X is not connected, it can be written as the disjoint union of nonvoid open sets V and W. Then these sets are also closed, hence compact and

$$\varepsilon_0 := \inf \{d(v, w),\ v \in V, w \in W\} > 0.$$

Hence for $\varepsilon < \varepsilon_0/2$ there cannot exist (ε, T)-chains from an element of V to an element of W. ■

We obtain the following characterization of the connected components of $\mathcal{R}$.

Theorem 5.8 *The connected components of the chain recurrent set $\mathcal{R}$ coincide with the maximal chain transitive subsets of $\mathcal{R}$. Furthermore, the flow restricted to a connected component of $\mathcal{R}$ is chain transitive.*

Proof. By Theorem 5.6 we know that the flow restricted to a maximal chain transitive subset $\mathcal{R}_0$ of $\mathcal{R}$ is chain transitive. Hence by the second part of Proposition 5.7 $\mathcal{R}_0$ is connected and thus contained in a connected component of $\mathcal{R}$. Conversely, the first part of Proposition 5.7 implies that every connected component of $\mathcal{R}$ is chain transitive, because it is closed, chain recurrent, and connected. Hence the first assertion follows. The second claim is an immediate consequence. ■

The connected components of $\mathcal{R}$ are called the *chain recurrent components*.

Example 5.9 *Consider again the dynamical system discussed in Examples 3.3 and 5.3. For this example, the components of the chain recurrent set, i.e. the chain recurrent components are $\{0\}$, $\{1\}$, $\{2\}$, and $\{3\}$.*

Example 5.10 *An example of a flow for which the limits sets from points are strictly contained in the chain recurrent components can be obtained as follows: Let $M = [0,1] \times [0,1]$. Let the flow Φ on M be defined such that all points on the boundary are fixed points, and the orbits for points $(x, y) \in (0,1) \times (0,1)$ are straight lines $\Phi(\cdot, (x, y)) = \{(z_1, z_2),\ z_1 = x,\ z_2 \in (0,1)\}$ with $\lim_{t\to\pm\infty} \Phi(t, (x, y)) = (x, \pm 1)$. For this system, each point on the boundary is its own α- and ω-limit set. The α-limit sets for points in the interior $(x, y) \in (0,1) \times (0,1)$ are of the form $\{(x, -1)\}$, and the ω-limit sets are $\{(x, +1)\}$. The only chain recurrent component for this system is $M = [0,1] \times [0,1]$, which is also the only Morse set.*

We also note the following simple lemma, which indicates a uniform upper bound for the total time needed to connect any two points in a chain recurrent component.

Lemma 5.11 *Let $\mathcal{R}_0$ be a chain recurrent component and fix ε, $T > 0$. Then there exists $\bar{T}(\varepsilon, T) > 0$ such that for all x, $y \in \mathcal{R}_0$ there is an (ε, T)-chain from x to y with total length $\leq \bar{T}(\varepsilon, T)$.*

Proof. By assumption, one finds for all x, $y \in \mathcal{R}_0$ an $(\frac{\varepsilon}{2}, T)$-chain from x to y. Using continuous dependence on initial values and compactness, one finds finitely many (ε, T)-chains connecting every $x \in \mathcal{R}_0$ with a fixed $z \in \mathcal{R}_0$. One also finds finitely many (modulo their endpoints) (ε, T)-chains connecting z with arbitrary elements $y \in \mathcal{R}_0$. Thus one ends up with finitely many (ε, T)-chains connecting all points in $\mathcal{R}_0$. The maximum of their total lengths is the desired upper bound $\bar{T}(\varepsilon, T)$. ■

6 Chain Recurrence and Attractors

We proceed to analyze the relation between chain recurrence and attractors, leading to the main result in Theorem 6.4.

Definition 6.1 *For $Y \subset X$ define the chain limit set*

$$\Omega(Y) = \left\{ z \in X, \quad \begin{array}{c} \text{there is } y \in Y \text{ such that for all } \varepsilon, T > 0 \\ \text{there is an } (\varepsilon, T)\text{-chain from } y \text{ to } z \end{array} \right\}.$$

Furthermore, for $\varepsilon, T > 0$ define

$$\Omega(Y, \varepsilon, T) = \{ z \in X, \text{ there are } y \in Y \text{ and an } (\varepsilon, T)\text{-chain from } y \text{ to } z \}.$$

One easily sees that $\omega(Y) \subset \Omega(Y)$.

Proposition 6.2 *For $Y \subset X$ the set $\Omega(Y)$ is the intersection of all attractors containing $\omega(Y)$.*

Proof. Note that $\Omega(Y) = \bigcap_{\varepsilon, T > 0} \Omega(Y, \varepsilon, T)$, and for $\varepsilon, T > 0$ define $N := \mathrm{cl}(\Omega(Y, \varepsilon, T))$. Then $\omega(N) \subset \Omega(Y, \varepsilon, T) \subset \mathrm{int}\, N$, where the second inclusion follows because $\Omega(Y, \varepsilon, T)$ is open and contained in N. Now let $z \in \omega(N)$. Then there are $t_n \to \infty$ and $x_n \in N$ with $x_n \cdot t_n \to z$. Choose $n_0 \in \mathbb{N}$, $\delta > 0$ and $p \in \Omega(Y, \varepsilon, T)$ with

$$d(p, x_{n_0}) < \delta, \; t_{n_0} > T, \text{ and } d(x_{n_0} \cdot t_{n_0}, z) < \frac{\varepsilon}{2},$$
$$d(z_{n_0} \cdot t_{n_0}, x_{n_0} \cdot t_{n_0}) < \frac{\varepsilon}{2} \text{ for all } z \text{ with } d(z, x_{n_0}) < \delta.$$

By definition of p there is an (ε, T)-chain from some $y \in Y$ to p and we obtain

$$d(p \cdot t_{n_0}, z) \leq d(p \cdot t_{n_0}, x_{n_0} \cdot t_{n_0}) + d(x_{n_0} \cdot t_{n_0}, z) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus concatenation yields an (ε, T)-chain from y to z.

We have shown that $A := \omega(N)$ is a closed invariant set with neighborhood N, hence an attractor. By invariance of $\Omega(Y)$ we have $A = \omega\left(\mathrm{cl}(\Omega(Y, \varepsilon, T))\right) \supset \Omega(Y) \supset \omega(Y)$. Direct inspection shows that $\Omega(Y) = \omega(\Omega(Y))$ in fact equals the intersection of these attractors containing $\omega(Y)$.

Now suppose that A is any attractor containing $\omega(Y)$. Let V be a neighborhood of A disjoint from A^* and let $t > 0$ be such that $(\mathrm{cl}\, V \cdot t) \subset V$. Let

$$0 < \varepsilon < \inf\left\{d(y, z),\ y \in V \text{ and } z \notin \mathrm{cl}\, V \cdot t\right\}.$$

Choose $T > t$ such that $Y \cdot T \subset \mathrm{cl}(V \cdot t)$. Then every (ε, T)-chain from Y must end in V. Therefore, if $\omega(x) \subset A$, then also $\Omega(x) \subset A$ and hence $\Omega(Y)$ is the intersection of all attractors containing $\omega(Y)$. ■

This proposition implies, in particular, that a chain transitive flow has only the trivial attractor $A = X$, because for every $Y \subset X$ one has that $\Omega(Y) = X$.

Example 6.3 *Consider again the dynamical system discussed in Example 3.3. For this dynamical system we have for any subset $Y \subset [0, 3]$ that $\omega(Y) = \Omega(Y)$. The proof is a combination of Examples 3.3 and 5.3. But Example 5.10 shows that strict inclusion may hold.*

We obtain the following relation between the chain recurrent set and attractors.

Theorem 6.4 *The chain recurrent set $\mathcal{R}$ satisfies*

$$\mathcal{R} = \bigcap \left\{A \cup A^*,\ A \textit{ is an attractor}\right\}.$$

In particular, there exists a finest Morse decomposition $\{\mathcal{M}_1, ..., \mathcal{M}_n\}$ if and only if the chain recurrent set $\mathcal{R}$ has only finitely many connected components. In this case, the Morse sets coincide with the chain recurrent components of $\mathcal{R}$ and the flow restricted to every Morse set is chain transitive and chain recurrent.

Proof. If A is an attractor and $x \in X$, either $\omega(x) \subset A$ or $\omega(x) \subset A^*$. If $x \in \mathcal{R}$, then, by Proposition 6.2, x is contained in every attractor, which contains $\omega(x)$. Hence $x \in A \cup A^*$. Conversely, if x is in the intersection, then x is in every attractor containing $\omega(x)$. Hence $x \in \Omega(x)$, that is $x \in \mathcal{R}$.
If there exists a finest Morse decomposition, then the flow restricted to a corresponding Morse set must be chain transitive, hence the Morse sets are connected components of $\mathcal{R}$. Conversely, the connected components $\mathcal{M}_i$ of $\mathcal{R}$ define a Morse decomposition, because they are isolated invariant sets ordered by (2). In fact, this is the finest Morse decomposition: Using the characterization of Morse decompositions via increasing attractor sequences, one sees that a finer Morse decomposition would imply the existence of an attractor A such that $A \cap \mathcal{M}_i$ is a proper subset of $\mathcal{M}_i$ for some i, and hence this would be an attractor of the flow restricted to $\mathcal{M}_i$. This contradicts chain transitivity of $\mathcal{M}_i$.
■

Remark 6.5 *There are at most countably many attractors, cp. [1, Chapter 3, Proposition 8] or [21, Lemma 9.1.7.].*

Finally, we show chain transitivity of the flow restricted to a limit set.

Proposition 6.6 *If the flow is topologically transitive, then it is chain transitive. In other words, a flow restricted to an ω-limit set $\omega(x)$ with $x \in X$ is chain transitive.*

Proof. Because ω-limit sets are connected, it suffices by Proposition 5.7 to show that the flow restricted to $\omega(x)$ is chain recurrent. Define a flow $(y,t) \mapsto y \cdot t$ on $[-1,1]$ by the equation $\dot{y} = 1 - y^2$. On $X \times [-1,1]$ define a flow by $(x,y) \mapsto (x \cdot t, y \cdot t)$. Then $Z = \mathrm{cl}\ ((x,0)\cdot \mathbb{R})$ is a compact invariant set. By Theorem 6.4 the chain recurrent set contains all ω-limit sets, and hence the chain recurrent set of the flow restricted to Z is

$$\mathcal{R}(Z) = \omega^*(x) \times \{-1\} \cup \omega(x) \times \{1\}.$$

By Theorem 5.6 the flow restricted to $\mathcal{R}(Z)$ is chain recurrent and the connected components of $\mathcal{R}(Z)$ are chain transitive. Hence the flow restricted to $\omega(x)\times\{1\}$ and thus the flow restricted to $\omega(x)$ are chain recurrent. ■

7 Ergodic Theory for Chains

In this section we explain how the classical construction of invariant measures as occupation measures along trajectories can be generalized to the construction along chains.

Standard references for the ergodic theory of flows. are Katok and Hasselblatt [14], Mañé [15], and Nemytskii and Stepanov [17]; see also Pollicott [20]. Recall that a σ-algebra on a set X is a family $\mathcal{A}$ of subsets of X such that $X \in \mathcal{A}$, the complement of every $A \in \mathcal{A}$ is again in $\mathcal{A}$, and countable unions of elements in $\mathcal{A}$ are in $\mathcal{A}$; in particular, this implies that finite intersections of elements in $\mathcal{A}$ are in $\mathcal{A}$. For a metric space X the Borel σ-algebra is the smallest σ-algebra containing all open (and hence all closed) subsets of X; the elements of this σ-algebra are called Borel sets. A map $\mu : \mathcal{A} \to \mathbb{R}$ is a measure on a σ-algebra $\mathcal{A}$, if for every countable family $(A_i)_{i\in\mathbb{N}}$ of pairwise disjoint sets $A_i \in \mathcal{A}$

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

A probability measure is a measure with $\mu(X) = 1$ and $\mu(A) \geq 0$ for all A. For a flow Φ on a metric space X an invariant measure μ is a probability measure on the Borel σ-algebra of X such that

$$\mu(A) = \mu(\Phi_{-t}A) := \mu(\{x \in X,\ \Phi_t(x) \in A\}$$

for all $t \in \mathbb{R}$ and all Borel sets A. It suffices to require this condition for all open (or all closed) subsets $A \subset X$. In the following, let X be a compact metric space.

Then the probability measures coincide with the Radon probability measures, that is the continuous linear functionals μ from the space of continuous functions $C(X)$ to $\mathbb{R}$ with $\mu(\chi_X) = 1$ where $\chi_X(x) = 1$ for all $x \in X$, and $\mu(f) \geq 0$ for all $f \in C(X)$ with $f(x) \geq 0$. The support $\mathrm{supp}(\mu)$ of a measure μ is the smallest closed subset K of X such that $\mu(f) = 0$ if f vanishes on K. An invariant measure is called *ergodic* if

$$\mu(A\Delta(\Phi_{-t}A)) = 0 \text{ for all } t \in \mathbb{R} \text{ implies } \mu(A) = 0 \text{ or } \mu(A) = 1,$$

where for subsets $A, B \subset X$ the symmetric difference is denoted by $A\Delta B = A \setminus B \cup B \setminus A$. The set of invariant measures is convex and weakly compact and the extremal points are the ergodic measures; see [15, Proposition II.2.5] or [14, Lemma 4.1.10].

A classical construction due to Krylov-Bogolyubov yields invariant measures as occupation measures along trajectories. Given $x \in X$ and $T > 0$ define a continuous linear functional L on $C(X)$ by

$$Lf := \frac{1}{T}\int_0^T f(\Phi_t x)\, dt.$$

This defines a Radon probability measure ν on X. For every sequence $T_k \to \infty$ a subsequence of the corresponding measures ν_k converges weakly to a Radon probability measure μ_x on X. This measure is in fact invariant for the flow Φ (see, e.g., [17, Theorem VI.9.05]), and hence the set $\mathcal{M}_\Phi$ of invariant measures is nonempty. This construction can be generalized to obtain invariant measures via chains.

Let ζ be an (ε, T)-chain in X given by $n \in \mathbb{N},\ T_i \geq T,\ x_i \in X,\ i = 0, 1, ..., n$. Then a continuous linear functional L_ζ on $C(X)$ is defined by

$$L_\zeta f = \left(\sum_{i=0}^{n-1} T_i\right)^{-1} \sum_{i=0}^{n-1} \int_0^{T_i} f(\Phi_t x_i)\, dt.$$

For $i = 0, ..., n-1$, the map

$$f \mapsto \frac{1}{T_i}\int_0^{T_i} f(\Phi_t x_i)\, dt$$

defines a Radon probability measure ν_i on X. The measure ν corresponding to L_ζ is a convex combination of the ν_i, hence also a Radon probability measure. Now consider for $\varepsilon^k \to 0,\ T^k \to \infty$ a sequence of (ε^k, T^k)-chains ζ^k, given by $n^k \in \mathbb{N},\ T_i^k \geq T^k$, and $x_i^k \in X$ for $i = 0, ..., n^k - 1,\ k \in \mathbb{N}$. Define L^k for ζ^k as earlier with corresponding measure ν^k. As $k \to \infty$, a subsequence of (ν^k) denoted again by (ν^k), converges weakly to a Radon probability measure μ on X, i.e., we have for all $f \in C(X)$

$$\lim_{k\to\infty} \left(\sum_{i=0}^{n^k-1} T_i^k\right)^{-1} \sum_{i=0}^{n^k-1} \int_0^{T_i^k} f(\Phi_t x_i^k)\, dt = \int_X f\, d\mu. \tag{3}$$

Theorem 7.1 *Let $\Phi : \mathbb{R} \times X \longrightarrow X$ be a continuous dynamical system on the compact state space X. Then the measure μ defined in (3) is invariant under the flow, that is, for all $f \in C(X)$ it holds that*

$$\int_X f(x)\, d\mu = \int_X f(\Phi_\tau(x))\, d\mu \text{ for all } \tau \in \mathbb{R}.$$

Proof. This assertion is–as in the standard Krylov-Bogolyubov construction–seen as follows: For $\tau \in \mathbb{R}$ and all i, k

$$\begin{aligned}
&\left| \int_0^{T_i^k} f(\Phi_{t+\tau} x_i^k)\, dt - \int_0^{T_i^k} f(\Phi_t x_i^k)\, dt \right| \\
&\leq \left| \int_\tau^{T_i^k} f(\Phi_t x_i^k)\, dt + \int_{T_i^k}^{T_i^k+\tau} f(\Phi_t x_i^k)\, dt - \int_0^{T_i^k} f(\Phi_t x_i^k)\, dt \right| \\
&\leq 2\tau \max |f(x)|
\end{aligned}$$

Hence for all $\delta > 0$ and all $T^k > T > 0$ large enough, one has

$$\left| \left(\sum_{i=0}^{n^k-1} T_i^k \right)^{-1} \sum_{i=0}^{n^k-1} \left\{ \int_0^{T_i^k} \left[f(\Phi_{t+\tau} x_i^k) - f(\Phi_t x_i^k) \right] dt \right\} \right| < \delta,$$

proving the assertion. ∎

8 Chain Recurrence for Families of Dynamical Systems

In general limit sets, Morse sets and chain recurrent components do not depend continuously on system parameters, see, e.g., bifurcation scenarios like the pitchfork or Hopf bifurcation, or the discussions and results on control flows in [6]. However, an upper semicontinuity holds for chain transitive sets, which will be made precise in this section.

Consider a family of dynamical systems on a compact metric space X depending on a papameter $\alpha \in A \subset \mathbb{R}^k$ of the form

$$\Phi : A \times \mathbb{R} \times X \longrightarrow \mathbb{R} \times X \tag{4}$$

where A is a (path) connected set and Φ is continuous in all components. We need some properties of set valued maps Γ defined on a metric space A with nonvoid value sets in a metric space X; compare, e.g., Castaing and Valadier [5], Aubin and Frankowska [2], or Warga [27].

Definition 8.1 *A set valued map $\Gamma : A \to X$ is lower semicontinuous at $\alpha_0 \in A$ if for all $\varepsilon > 0$ there is $\delta > 0$ such that $d(\alpha, \alpha_0) < \delta$ implies that $\sup_{x \in \Gamma(\alpha_0)} d(x, \Gamma(\alpha)) < \varepsilon$. It is called upper semicontinuous at $\alpha_0 \in A$, if for all $\varepsilon > 0$ there is $\delta > 0$ such that $d(\alpha, \alpha_0) < \delta$ implies that $\sup_{x \in \Gamma(\alpha)} d(x, \Gamma(\alpha_0)) < \varepsilon$.*

Note that Γ is upper and lower semicontinuous if and only if it is continuous with respect to the Hausdorff metric (1). Furthermore, if Γ has compact values, lower semicontinuity is equivalent to

$$\Gamma(\alpha_0) \subset \liminf_{\alpha\to\alpha_0} \Gamma(\alpha) := \left\{ \begin{array}{c} x \in X, \text{ for all } \alpha_k \to \alpha \text{ in } A \\ \text{there are } x_k \in \Gamma(\alpha_k) \text{ with } x_k \to x \end{array} \right\}.$$

Upper semicontinuity is equivalent to

$$\Gamma(\alpha_0) \supset \limsup_{\alpha\to\alpha_0} \Gamma(\alpha) := \left\{ \begin{array}{c} x \in X, \text{ there are } \alpha_k \to \alpha \text{ in } A \\ \text{and } x_k \in \Gamma(\alpha_k) \text{ with } x_k \to x \end{array} \right\}.$$

The following theorem shows that maximal (with respect to set inclusion) chain transitive subsets $Y \subset X$ depend upper semicontinuously on $\alpha \in A$.

Theorem 8.2 *Consider the parameter dependent system (4). For a sequence $\alpha_k \to \alpha_0$ in A consider maximal chain transitive sets $E^{\alpha_k} \subset X$ of $(4)^{\alpha_k}$. Then there exists a maximal chain transitive set E^{α_0} of $(4)^{\alpha_0}$ such that*

$$\limsup{}_{\alpha_k\to\alpha_0} E^{\alpha_k} := \{x \in X, \text{ there are } x^{\alpha_k} \in E^{\alpha_k} \text{ with } x^{\alpha_k} \to x\} \subset E^{\alpha_0}.$$

Of course, the set on the left-hand side of this inclusion may be empty, in which case the statement is trivial.

Proof. Pick y^1, y^2 in $\limsup_{\alpha_k\to\alpha_0} E^{\alpha_k}$. We have to show that y^1 and y^2 are in some chain transitive set of $(4)^{\alpha_0}$. Let ε, $T > 0$. We will construct an (ε, T)-chain from y^1 to y^2. For $i = 1, 2$, one has $y^i = \lim_{k\to\infty} x_k^i$ with $x_k^i \in E^{\alpha_k}$, $k \in \mathbb{N}$. For all $k \in \mathbb{N}$ there are $(\frac{\varepsilon}{3}, T)$-chains from x_k^1 to x_k^2, i.e., there are $n_k \in \mathbb{N}$, $z_0^k, ..., z_{n_k}^k \in X$ and $t_0^k, ..., t_{n_k-1}^k \geq T$ with $z_0^k = x_k^1$, $z_{n_k}^k = x_k^2$ and

$$d(\Phi^{\alpha^k}(t_j^k, z_j^k), z_{j+1}^k) < \frac{\varepsilon}{3} \text{ for } j = 0, 1, ..., n_k - 1. \tag{5}$$

Using compactness of X and continuity of the family Φ^α one finds $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, all $z \in X$, and all $0 \leq t \leq 2T$

$$d(\Phi^{\alpha_0}(t, z), \Phi^{\alpha_k}(t, z)) < \frac{\varepsilon}{3}. \tag{6}$$

We may choose k_0 so large that we also have for $k \geq k_0$

$$d(\Phi^{\alpha_0}(T, y^1), \Phi^{\alpha_k}(T, x_k^1)) < \frac{\varepsilon}{3} \tag{7}$$

and

$$d(x_k^2, y^2) < \frac{\varepsilon}{3}. \tag{8}$$

In the following, we will fix $k \geq k_0$ and drop the index k everywhere except in α_k. Define an (ε, T)-chain for α_k from y^1 to y^2 in the following way. The points are

$$\begin{array}{c} y_{0,0} = y^1,\ y_{0,1} = \Phi^{\alpha_k}(T, x^1), \text{ and for } j = 1, 2, ..., n-1 \\ y_{j,i} = \Phi^{\alpha_k}(iT, z_j),\ i = 0, 1, ..., i_j, \text{ and } y_n = y^2, \end{array} \tag{9}$$

where $i_j \in \mathbb{N}$ is such that $(i_j+1)T \le t_j < (i_j+2)T$; and the times are

$$\begin{aligned} t_{0,0} = T, \text{ and for } j = 1,2,...,n-1 \\ t_{j,i} = T\,, \ \ i = 0,1,...,i_j-1, \ t_{j,i_j} = t_j - i_jT. \end{aligned} \tag{10}$$

In fact, this is an (ε, T)-chain from y^1 to y^2 with $1 + \sum_{j=1}^{n-1} i_j$ jumps of size less than ε:

$$d(\Phi^{\alpha_0}(t_{0,0}, y_{0,0}), y_{0,1}) = d(\Phi^{\alpha_0}(T, y^1), \Phi^{\alpha_k}(T, x^1)) < \varepsilon$$

by (7); for $i = 0,1,...,i_j-1, \ j = 1,...,n-1$

$$\begin{aligned} & d(\Phi^{\alpha_0}(t_{j,i}, y_{j,i}), y_{j,i+1}) \\ &= d(\Phi^{\alpha_0}(T, \Phi^{\alpha_k}(iT, z_j)), \Phi^{\alpha_k}((i+1)T, z_j)) \\ &= d(\Phi^{\alpha_0}(T, \Phi^{\alpha_k}(iT, z_j)), \Phi^{\alpha_k}(T, \Phi^{\alpha_k}(iT, z_j))) \\ &< \varepsilon \end{aligned}$$

by (6); for $i = i_j, \ j = 0,1,...,n-1$

$$\begin{aligned} & d(\Phi^{\alpha_0}(t_{j,i_j}, y_{j,i_j}), y_{j+1,0}) \\ &= d(\Phi^{\alpha_0}(t_j - i_jT, \Phi^{\alpha_k}(i_jT, z_j)), z_{j+1}) \\ &\le d(\Phi^{\alpha_0}(t_j - i_jT, \Phi^{\alpha_k}(i_jT, z_j)), \Phi^{\alpha_k}(t_j - i_jT, \Phi^{\alpha_k}(i_jT, z_j))) \\ &\quad + d(\Phi^{\alpha_k}(t_j, z_j), z_{j+1}) \\ &< \varepsilon \end{aligned}$$

by (6) and (5). Finally, for $j = n-1$ and $i = i_j = i_{n-1}$

$$\begin{aligned} & d(\Phi^{\alpha_0}(t_{j,i}, y_{j,i}), y_{j+1}) \\ &= d(\Phi^{\alpha_0}(t_{n-1} - i_{n-1}T, y_{n-1,i_{n-1}}), y_n) \\ &= d(\Phi^{\alpha_0}(t_{n-1} - i_{n-1}T, \Phi^{\alpha_k}(i_{n-1}T, z_{n-1})), y^2) \\ &\le d(\Phi^{\alpha_0}(t_{n-1} - i_{n-1}T, \Phi^{\alpha_k}(i_{n-1}T, z_{n-1})), \\ &\quad \Phi^{\alpha_k}(t_{n-1} - i_{n-1}T, \Phi^{\alpha_k}(i_{n-1}T, z_{n-1}))) \\ &\quad + d(\Phi^{\alpha_k}(t_{n-1}, z_{n-1}), x^2) + d(x^2, y^2) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

by (6), (5), and (8). ■

In specific situations, stronger results are valid. One such situation is given by a one-parameter family $(4)^\alpha$ with $\alpha \in A \subset \mathbb{R}$ and an increasing family of chain transitive sets. This situation is common in the theory of control flows, compare, e.g., [6]. Indeed, the following, more general result holds.

Proposition 8.3 *Let Γ be a set valued map defined on a real interval $[\alpha_*, \alpha^*)$, $0 \le \alpha_* < \alpha^* \le \infty$, with compact values in a compact metric space X and suppose that Γ is monotonically increasing, that is,*

$$\Gamma(\alpha) \subset \Gamma(\alpha') \ \text{if}\, \alpha \le \alpha'.$$

Then Γ is continuous (with respect to the Hausdorff metric) at all but at most countably many points $\alpha_0 \in [\alpha_*, \alpha^*)$.

Proof. Let $\{x_n,\ n \in \mathbb{N}\}$ be a countable dense subset of X. Then for every $n \in \mathbb{N}$ the map $\alpha \mapsto c_n(\alpha) := d(x_n, \Gamma(\alpha))$ is monotonically decreasing, hence it has at most countably many points ρ_n^m, $m \in \mathbb{N}$, of discontinuity (see Natanson [16] or Hewitt and Stromberg [12]). Thus it is sufficient to show that every point α_0 of discontinuity of Γ is also a point of discontinuity for some c_n. Then the countable set $\{\alpha_n^m,\ n, m \in \mathbb{N}\}$ contains all points of discontinuity of Γ. Let first $\alpha > \alpha_0$ and consider

$$d_H(\Gamma(\alpha), \Gamma(\alpha_0)) = \sup_{x \in \Gamma(\alpha)} d(x, \Gamma(\alpha_0)).$$

If $\lim_{\alpha_k \searrow \alpha_0} d_H(\Gamma(\alpha_k), \Gamma(\alpha_0)) =: 3\varepsilon_0 > 0$, then there is for all k large enough a point $y_k \in \Gamma(\alpha_k) \subset \Gamma(\alpha_m)$ for $m \leq k$ with $d(y_k, \Gamma(\alpha_0)) \geq 3\varepsilon_0$. Every cluster point y of this sequence satisfies $y \in \Gamma(\alpha_k)$ for all k and $d(y, \Gamma(\alpha_0)) \geq 3\varepsilon_0$. Then there is a point x_n with $c_n(\alpha_0) = d(x_n, \Gamma(\alpha_0)) \geq 2\varepsilon_0$ and $d(x_n, y) \leq \varepsilon_0$; hence $c_n(\alpha_k) = d(x_n, \Gamma(\alpha_k)) \leq \varepsilon_0$ for all k. Thus c_n is discontinuous at $\alpha = \alpha_0$. For $\alpha_k \nearrow \alpha_0$ one argues similarly. ■

Remark 8.4 *Proposition 8.3 is known as Scherbina's Lemma; see Pilyugin [19, Lemma 4.1.3] and Scherbina [26]. Scherbina's Lemma [19] states that increasing, compact-valued mappings defined on $[0, \infty)$ are continuous with respect to the Hausdorff metric at all but countably many $\alpha-$values*

Remark 8.5 *The preceding elementary proof is based on the classical fact that real-valued monotonically increasing maps have at most countably many points of discontinuity (Carathéodory [4], § 158, p.154), which, in turn, is based on the elementary fact that an uncountable sum of positive numbers cannot be finite ([22], p. 38). See, however, [4], § 156, for an example, where these points of discontinuity are everywhere dense.*

References

[1] E. Akin, The General Topology of Dynamical Systems, Amer. Math. Soc., Providence, R.I., 1993.

[2] J.-P. Aubin and H. Frankowska, Set Valued Analysis, Birkhäuser, 1990.

[3] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey, On Devaney's definition of chaos, Amer. Math. Monthly 99 (1992), 332-334.

[4] C. Carathéodory, Vorlesungen über Reelle Funktionen, Leipzig, 1918. Reprint: Chelsea Publ. Co., 1968.

[5] C. Castaing and M. Valadier, Convex Analysis and Measurable Multifuntions, LN Mathematics vol. 580, Springer-Verlag, 1977.

[6] F. Colonius and W. Kliemann, The Dynamics of Control, Birkhäuser, 2000.

[7] C. Conley, Isolated Invariant Sets and the Morse Index, CBMS Regional Conference Series no. 38, Amer. Math. Soc., 1978.

[8] C. Conley, The gradient structure of a flow I, Ergoc Theory Dyn. Syst. 8 (1988), 11-26. Original: IBM Research, RC 3939, Yorktown Heights, N.Y., 1972.

[9] R. Devaney, An Introduction to Chaotic Dynamical Systems, Benjamin/Cummings, 1986.

[10] R. Easton, Geometric Methods for Discrete Dynamical Systems, Oxford University Press, 1998.

[11] F. Engelking, General Toplogy, PWN-Polish Scientific, Warszawa, 1977.

[12] E. Hewitt and K. Stromberg, Real and Abstract Analysis, Springer-Verlag, 1965.

[13] M. Hurley, Chain recurrence, semiflows, and gradients, J. Dyn. Diff. Equations 7 (1995), 437-456.

[14] A. Katok and B. Hasselblatt, An Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, 1995.

[15] R. Mañé, Ergodic Theory and Differentiable Dynamics, Springer-Verlag, 1987.

[16] V.I. Natanson, Theorie der Funktionen einer rellen Veränderlichen, Second Edition, Akademie-Verlag, Berlin, 1961. Russian original 1957.

[17] V.V. Nemytskii and V.V. Stepanov, Qualitative Theory of Dynamical Systems, Princeton University Press, 1960. Russian original 1949.

[18] G.K. Pedersen, Analysis Now, Springer-Verlag, 1988.

[19] S. Pilyugin, The Space of Dynamical Systems with the C^0-Topology, Springer-Verlag, 1994.

[20] M. Pollicott, Lectures on Ergodic Theory and Pesin Theory of Compact Manifolds, London Mathematical Society Lect. Notes Series 180, Cambridge University Press, 1993.

[21] C. Robinson, Dynamical Systems. Stability, Symbolic Dynamics, and Chaos, CRC Press Inc., 1995.

[22] H.L. Royden, Real Analysis, Second Edition, Macmillan Publishing Co., 1968.

[23] K.P. Rybakowski, The Homotopy Index and Partial Differential Equations, Springer-Verlag, 1987.

[24] D. Salamon, Connected simple systems and the Conley index of isolated invariant sets, Trans. Amer. Math. Soc. 291 (1985), 1-41.

[25] D. Salamon and E. Zehnder, Flows on vector bundles and hyperbolic sets, Trans. Amer. Math. Soc. 306 (1988), 623-649.

[26] N. Scherbina, Continuity of one-parameter families of sets, Dokl. Ak. Nauk SSSR 234 (1977), 327-329. In Russian.

[27] J. Warga, Optimal Control of Differential and Functional Equations, Academic Press, 1972.