ITERATING ANALYTIC SELF-MAPS OF DISCS AND APPLICATIONS TO DYNAMICAL SYSTEMS

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ABSTRACT. Möbius or linear fractional transformations are a specific type of complex conformal map and have very interesting properties. We want to analyze its iterative properties given this analytic, self-map. Applications of other iterative systems are also explored.

1. INTRODUCTION

We want to investigate a specific type of complex mapping that has some unique and interesting properties in the complex plane. We wanted to see how this transformation worked, its long-term behavior through many iterations and how it could be applied to problems. The mapping that we looked at is called a *linear fractional transformation* or *Möbius transformation*, it is defined by: (Note: D denotes the complex unit-disk, and $\overline{D}$ denotes $D \cup \partial D$, where ∂D is the boundary of the complex unit-disk, also C denotes the set of all complex numbers. A complex number $z = x + iy$ has a complex conjugate, denoted $\bar{z}$, where $\bar{z} = x - iy$).
For each $a \in D$, and $u \in \partial D$

$$T(z) = uT_a(z) = u\frac{z-a}{1-\bar{a}z}, \text{ for all } z \in C \text{ if } a = 0 \text{ and}$$
$$\text{all } z \in C \setminus \{1/\bar{a}\} \text{ for } a \neq 0 \text{ such that } |z| \leq 1.$$

Below are some useful definitions, theorems, and lemmas presented without proof for use later on. Also below are proofs to show that this map is a conformal self-map, meaning it's one-to-one, onto, and holomorphic (or analytic) along with sending the domain $\overline{D} \to \overline{D}$ and $\partial D \to \partial D$.

Definition 1.1: A complex-valued function T is called *analytic* or *holomorphic* on an open set G if there is a derivative at every point in G, or can be written as a power series around a point z_0 in the form $T(z) = a_0 + \sum_{n=1}^{\infty} a_n (z - z_0)^n$.

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0.1 *Uniqueness Theorem* – If f and g are holomorphic in the open, connected set Θ and $f = g$ in some nonempty open subset of Θ, then $f = g$ throughout Θ.

0.2 *Open Map Theorem* – The image of a region under any nonconstant holomorphic function is an open set.

0.3 *Schwarz's Lemma* – If $f : D \to D$ is holomorphic and $f(0) = 0$, then $|f(z)| \leq |z|$ for all $z \in D$.

0.4 *Weierstrass's Theorem* – A locally uniform limit of holomorphic functions is a holomorphic function.

0.5 *Montel's Theorem* – Every uniformly bounded family of holomorphic functions in a region is a normal family; that is, any sequence in it contains a subsequence which is locally uniformly convergent throughout the region. (The limit is holomorphic by 0.3.)

0.6 *Corollary to Montel's Theorem* – If $\{f_n\}$ is a uniformly bounded sequence of holomorphic functions in a region Θ and if every convergent subsequence of $\{f_n\}$ has the same limit, then the sequence $\{f_n\}$ is convergent.

Proof of 1-to-1 property: Proof by contradiction - Show if $T(z_1) \neq T(z_2)$, then $z_1 \neq z_2$.

$$u\frac{z_1 - a}{1 - \bar{a}z_1} \neq u\frac{z_2 - a}{1 - \bar{a}z_2} \quad \Rightarrow \quad (z_1 - a)(1 - \bar{a}z_2) \neq (z_2 - a)(1 - \bar{a}z_1)$$

$$z_1 - \bar{a}z_1z_2 - a + a\bar{a}z_2 \neq z_2 - \bar{a}z_1z_2 - a + a\bar{a}z_1$$

$$z_1 - a\bar{a}z_1 \neq z_2 - a\bar{a}z_2 \quad \Rightarrow \quad z_1(1 - a\bar{a}) \neq z_2(1 - a\bar{a}) \quad \Rightarrow \quad z_1 \neq z_2$$

■

Proof of onto property: Use $w = T(z) = u\dfrac{z - a}{1 - \bar{a}z}$, and solve for z.

$$w(1 - \bar{a}z) = u(z - a) \quad \Rightarrow \quad z(u + \bar{a}w) = w + ua \quad \Rightarrow \quad z = u\frac{w + au}{1 + \bar{a}\bar{u}w}.$$

■

Proof of holomorphic/analytic property: Use the Neumann series $\left(\dfrac{1}{1 - z} = \sum_{n=0}^{\infty} z^n\right)$ to expand our given T function into a power series. Doing some algebra yields

$$T(z) = u(z - a)(1 + \frac{z_0\bar{a}}{1 - z_0\bar{a}})\sum_{n=0}^{\infty}(\bar{a}(1 - \frac{z_0\bar{a}}{1 - z_0\bar{a}}))^n (z - z_0)^n.$$

■

Theorem 1.1: For each $a \in D$ the function T_a maps $\overline{D}$ one-to-one onto $\overline{D}$, carrying D onto D and $T_a^{-1} = T_{-a}$. Conversely, any conformal map of D onto D has the form uT_a for some $a \in D, u \in \partial D$.

Proof: One calculates that for any $z \in \overline{D}, a \in D$,

$$\left|T_a(z)\right|^2 = 1 - \frac{(1-|z|^2)(1-|a|^2)}{\left|1-\overline{a}z\right|^2}.$$

This shows at once that T_a maps $\overline{D}$ onto $\overline{D}$, D onto D, and ∂D into ∂D. Consequently, $T_a \circ T_{-a}$ and $T_{-a} \circ T_a$ can be formed. A simple calculation reveals that each equals the identity function on $\overline{D}$.

If F is a conformal map of D onto D, let $a = F^{-1}(0)$ and consider $f = F \circ T_a^{-1}$. This is again a conformal map of D onto D, by the result of the first paragraph. Since $f(0) = 0$, we can apply Schwarz's Lemma to both f and f^{1} to get

$$|z| = |f^{-1}(f(z))| \le |f(z)| \le |z|, \qquad \forall z \in D.$$

The holomorphic function $f(z)/z$ thus has constant modulus 1 and so is constant, by the Open Map Theorem. Calling this unimodular constant u, we have $f = uI$, so $F = f \circ T_a = uT_a$.

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In general, this transformation is a rational function in the following form:

$$T(z) = \frac{az+b}{cz+d}$$

Using this general form we can define a matrix A such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We see that for our example that $A = \begin{pmatrix} u & -ua \\ -\overline{a} & 1 \end{pmatrix}$. This set of transformations form a group under matrix multiplication. The formal statement of the theorem and proof is given below.

Theorem 1.2: The set of all Möbius transformations or linear fractional transformations Ω form a group under matrix multiplication.

Proof: *i*) Closure – Given two matrices $A = \begin{pmatrix} u & -ua \\ -\overline{a} & 1 \end{pmatrix}, B = \begin{pmatrix} u & -ub \\ -\overline{b} & 1 \end{pmatrix}$ such that $A, B \in \Omega$, then show $AB \in \Omega$. Simply compute the product and we see that

$$AB = \begin{pmatrix} u^2 & -u^2ab \\ -\overline{a}\overline{b} & 1 \end{pmatrix}.$$

Plug this back into the entries of the general form of T and we see that

$$T(z) = \frac{az+b}{cz+d} = \frac{u^2 z - u^2 ab}{1-\bar{a}\bar{b}z} = u^2\,\frac{z-ab}{1-\bar{a}\bar{b}z}.$$

Which is the same form of $T(z)$, so $AB \in \Omega$, thus Ω is closed. *ii*) Associative – Obvious, matrix multiplication is associative. *iii*) Identity element - $I = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$. Plugging the entries into $T(z)$ we see that $T(z) = \frac{az+b}{cz+d} = \frac{uz}{1} = uz = z$, so Ω has a unique identity element. *iv*) Inverse element - $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ such that $\det A \neq 0$, so Ω has a unique inverse element. Therefore Ω is a group.

■

Theorem 1.1 shows that this transformation maps all points $z \in D$ to the open disk, and all boundary points to the boundary. We are going to use this point extensively later on. Similarly if given an open disc in *D* with a center *c* and radius *r* so that $\{z \in C : |z-c| < r\}, c \in C, r \geq 0$, (denoted $D(c,r)$) then this transformation maps every disc lying in *D* onto another disc in *D*. This concept is stated and proved in the following theorem and proof.

Theorem 1.3: For each $a \in D, 0 \leq r < 1$ the set $\{z \in C : |T_a(z)| \leq r\} = T_a^{-1}(\overline{D}(0,r))$ is the closed disc with the center $C^* = (1-r^2)a/(1-|a|^2 r^2)$ and radius $R = (1-|a|^2)r/(1-|a|^2 r^2)$, and it also lies in *D*.

Proof: By multiplying out everything, one sees first that

$$|T_a(z)|^2 \leq r^2 \Leftrightarrow (1-|a|^2 r^2)|z|^2 + 2(r^2-1)\operatorname{Re}(\bar{a}z) \leq r^2 - |a|^2$$

and the by "completing the square," that the latter inequality is equivalent to $|z-C^*|^2 \leq R^2$. Moreover, calculation shows that

$$|C^*| + R = 1 - \frac{(1-|a|)(1-r)}{1+|a|r} < 1$$

and therefore $D(C^*, R)$ lies in *D*.

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An important and useful representation of $T_a(z)$ is provided in the nest result, however no proof is needed since it is a straightforward algebraic simplification.

Theorem 1.4: Let $a \in D, u \in \partial D, T(z) = uT_a(z), and \quad z_1, z_2 \in \overline{D}$. Then

$$\frac{T(z)-T(z_1)}{T(z)-T(z_2)} = \frac{1-\overline{a}z_2}{1-\overline{a}z_1} \cdot \frac{z-z_1}{z-z_2} \qquad \forall z \in D \setminus \{z_2\}.$$

If z_1, z_2 are distinct fixed points of *T* and if we write λ for $(1-\overline{a}z_2)/(1-\overline{a}z_1)$ and $S(z) = (z-z_1)/(z-z_2) \quad (z \in C \setminus \{z_2\})$, then the last theorem tells us that $S(T(z)) = \lambda S(z)$ this implies that $T(z) = S^{-1}(\lambda S(z))$ for all $z \in D \setminus \{z_2\}$.
Iterating this function *T* *n*-times yields the following relation:

$$T^n(z) = S^{-1}(\lambda^n S(z)).$$

It will be shown later that this map *T* either yields one fixed point, or two fixed points either both on the boundary or one in the circle and one out. Similarly for one fixed point z_0 it can be shown that

$$S(T(z)) = \frac{z_0}{T(z)-z_0} = \frac{u-1}{u+1} \cdot \frac{z_0}{z-z_0} = \lambda S(z) \quad \text{for all } z \in D.$$

2. GENERALIZATIONS of $T(z)$ and CONJUGACY

This next definition of $T(z)$ generalizes the properties of $T(z)$ and its fixed points:

Definition 2.1:

i) $T(z)$ is *elliptic* if there is one fixed point $z_1 \in D$, and one fixed point $z_2 = \frac{1}{\overline{z}_1} \notin D$.

ii) $T(z)$ is *hyperbolic* if there are two fixed points $z_1, z_2 \in \partial D$

iii) $T(z)$ is *parabolic* if there is one fixed point $z_1 \in \partial D$.

This theorem shows how all 3 cases are satisfied.

Theorem 2.1: For each $a \in D, u \in \partial D$ the map $T(z)$ of $\overline{D}$ onto $\overline{D}$ either is the identity map or has one or two fixed points.

Proof: Suppose $T(z) \neq I$. The statement $T(z) = z$ for some $z \in \overline{D}$ is equivalent to

$$\overline{a}z^2 + (u-1)z - ua = 0 \qquad (*)$$

If *a* = 0, then $u \neq 1$ (since $T(z) \neq I$) and there is exactly one $z \in C$ which satisfies (*), namely, *z* = 0. Now suppose $a \neq 0$. Then *z* = 0 is not a root of (*) and, remembering that $\overline{u} = 1/u$, we see that for any $z \neq 0$

$$\overline{a}\left(\frac{1}{\overline{z}}\right)^2 + (u+1)\left(\frac{1}{\overline{z}}\right) - ua = -\frac{u}{\overline{z}^2}\overline{\left[\overline{a}z^2 + (u-1)z - ua\right]}.$$

So we can see now that (*) has one or two roots. We can also see that if 2 roots exist, then they are either both on the boundary of the disk, or one is in the disk and one is not in the disk.

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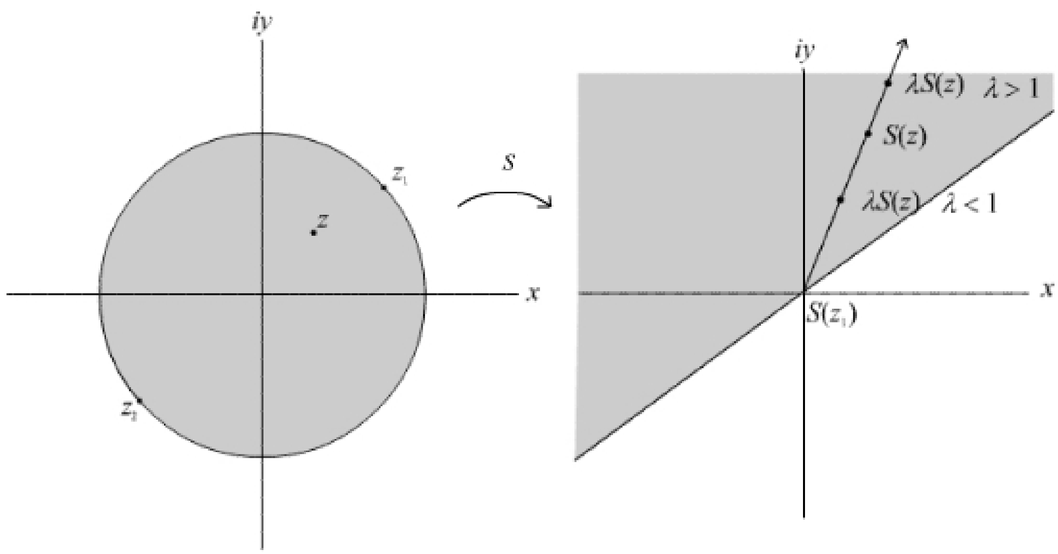
By using these definitions and using some analysis, we see that the parabolic case is the limiting case of both the elliptic and hyperbolic case.

We can now also describe some of the properties of λ. Here is a theorem describing some of the properties of λ for two fixed points.

Theorem 2.2: If $a \in D, u \in \partial D$ and $T(z)$ is hyperbolic then λ is not unimodular, real, and greater than zero.

Proof: For j = 1,2, the equation $z_j = uT_a(z_j) = u(z_j - a)/(1-\bar{a}z_j)$ implies $1-\bar{a}z_j = u(z_j - a)/z_j = u(1-(a/z_j)) = u(1-a\bar{z}_j)$. It follows that $\lambda = \bar{\lambda}$, so λ is real. Since $z_1 \neq z_2$, it is clear that $\lambda - 1 = \bar{a}(z_1 - z_2)/(1-\bar{a}z_1) \neq 0$. Also $|\bar{a}(z_1+z_2)| \leq |a|(|z_1|+|z_2|) = 2|a| < 2$, so $\lambda + 1 = [2-\bar{a}(z_1+z_2)]/(1-\bar{a}z_1) \neq 0$. So $|\lambda| \neq 1$. To show λ is positive, we use that fact $T^n(z) = S^{-1}(\lambda^n S(z))$, and that S maps "circles" to "circles". I put circles in quotes because of the fact that in the complex plane if you extend a circles' radius out to infinite it can look like a straight line. We simply have to look at the $\lambda^n S(z)$ part of the T-expression above. (See figure below)



Now, if $\lambda < 0$, then $\lambda S(z)$ would be reflected about the orgin into the white area – this is not possible since the inside of the circle on the left has to be mapped to the inside on the right. So $\lambda > 0$.

■

This expression of T $[T^n(z) = S^{-1}(\lambda^n S(z))]$ brings out a very interesting tool to help us do analysis on T. This concept is called *conjugacy*. Given two dynamical systems

(X,T), and (Y,R) (Note: This notation means a dynamical system with spaces X, Y respectively with mappings T and R.) We can use the expression for T and the example of our dynamical system to define the following:

$$(S \circ T \circ S^{-1})(z) = \lambda z = R(z).$$

Using this definition we can construct a commutative diagram to illustrate the mappings:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "S"'] & X \arrow[d, "S"] \\ Y \arrow[r, "R"'] & Y \end{tikzcd}$$

You can see from the diagram that this next expression holds true:

$$S \circ R = T \circ S.$$

Using the concept of conjugacy we can analyze T by analyzing a much simpler maps S using given or finding a function R. All of which have z as the variable of interest.

This next theorem shows how closely related the elliptic case is to a rotation.

Theorem 2.3: If $T(z)$ is elliptic, then it is conjugate to a rotation.

Proof: Let z_1 be the fixed point in D, Define $T_{z_1}(z) = \dfrac{z - z_1}{1 - \bar{z}_1 z}$. We see that this maps $z_1 \to 0$, and that $T_{z_1} \circ T \circ T_{z_1}^{-1}$ maps $0 \to z_1 \to z_1 \to 0$, so 0 is mapped to 0, thus $T(z)$ is a rotation.

■

We now have all the properties to do some long-term behavior analysis on T. Looking at the two fixed point cases, we found that $S(z) = \dfrac{z - z_1}{z - z_2}$. We can also see that $S(z_1) = 0$, and $S(z_2) = \infty$. When we look at λ, two cases arise – $0 < \lambda < 1$ or $\lambda > 1$. If $0 < \lambda < 1$, then $\lim_{n\to\infty} \lambda^n S(z_0) = S(z_1) = 0$. So $T^n(z) = S^{-1}(0) = z_1 \in \partial D$. Also if $\lambda > 1$, then $\lim_{n\to\infty} \lambda^n S(z_0) = S(z_2) = \infty$. So $T^n(z) = S^{-1}(\infty) = z_2 \in \partial D$. So if $\lambda < 1$, then z_1 is considered an attracting fixed point and z_2 is a repelling fixed point. For $\lambda > 1$, then z_2 is considered an attracting fixed point and z_1 is a repelling fixed point. These are very interesting and important facts, and will be shown in the diagrams below.

3. NUMERICAL COMPUTATION and DIAGRAMS

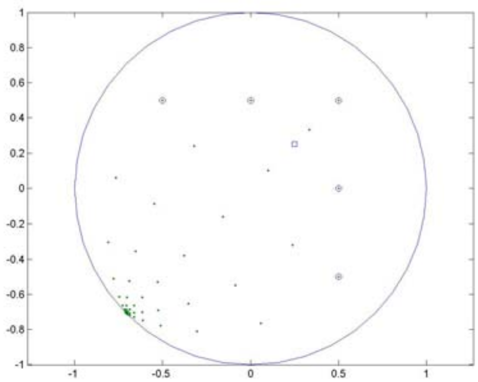
To see this for ourselves, I constructed a MATLAB script to run T 100 iterations given values for a, a starting point z_0 (z in the script), and u. Here is the script:

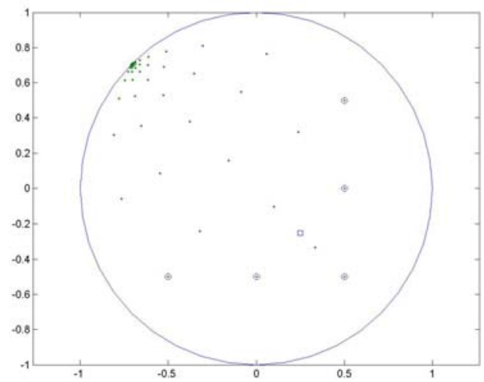
```
function Z = evaluate

format long

z=.5 + .5i;
n=100;
a=.05 + .0000000001i;
u=.7071 + .7071i;

a_real = real(a);
a_imag = imag(a);
a_length = sqrt((a_real)^2 + (a_imag)^2);

if a_length > 1
    error('a is outside complex unit circle - choose another a inside')
end

z_real = real(z);
z_imag = imag(z);
z_length = sqrt((z_real)^2 + (z_imag)^2);

if z_length > 1
    error('z is outside complex unit circle - choose another z inside')
end

k=1;                                  %initialize the variable k
ztemp=z;                              %initialize the input variable
Z=zeros(n,1);                         %create an n-vector of zeros
A=eye(n);                             %create the nxn identity matrix
Z=z*A(:,1);
while k<=(n-1)
    e=A(:,k+1);                         %define the standard unit vector
e_k
    x = u*((ztemp-a)/(1-((conj(a))*ztemp)));  %evaluate
    ek=x*e;
    Z=Z+ek;
    ztemp=x;
    k=k+1;
end

t = 0:.0001:2*pi;                      %set parameter for unit circle plot
plot(sin(t),cos(t),real(Z),imag(Z),'.')  %plot unit circle and iterates
hold on
plot(a,'s')
plot(z,'o')
plot(u,'d')
axis equal
```

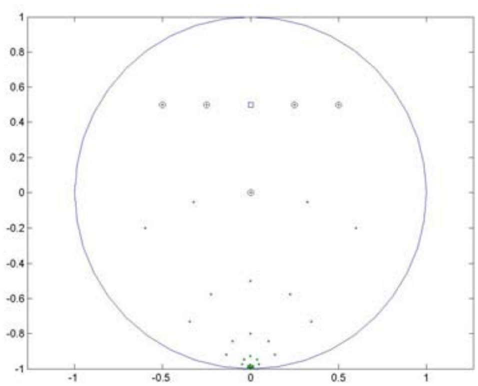
We can see by looking at the pictures that iterating *T* enough times using an *a* in the disk, a *u* on the boundary, and a starting point z_0 will eventually converge to the boundary in the hyperbolic and parabolic cases. However, in the elliptic case, it was shown above that this case is conjugate to a rotation. It can be shown that this collection of iterative points in the elliptic case creates an "average" circle in *D*. Some pictures below will make this clear.

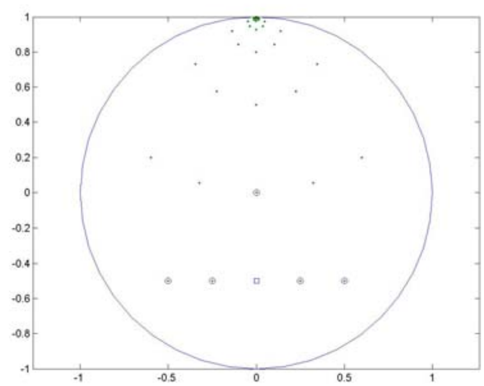
(Note: a is shown as a square, the starting points are in circles, and u is a diamond in the later pictures)

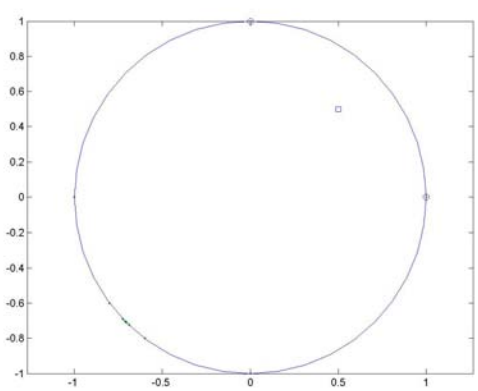
Here are some of the pictures produced if $u = (1,0i)$ (i.e. - hyperbolic cases).

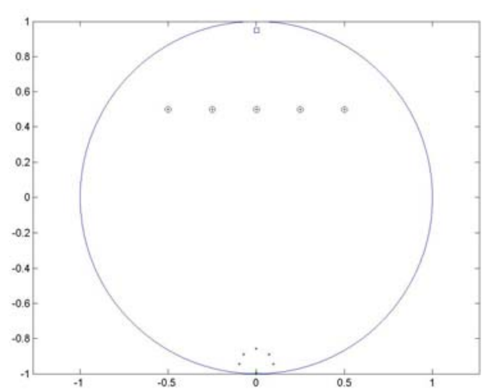


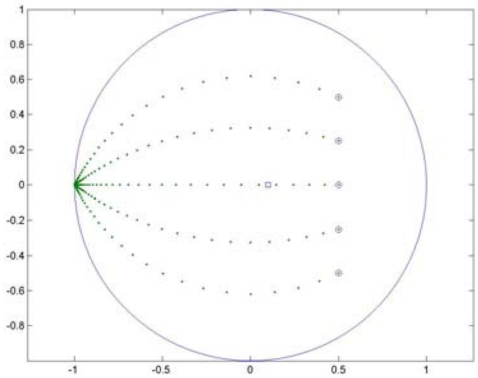


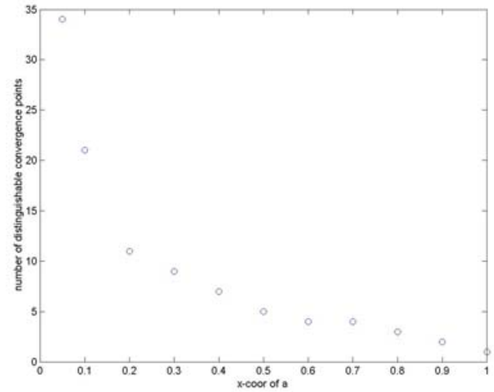










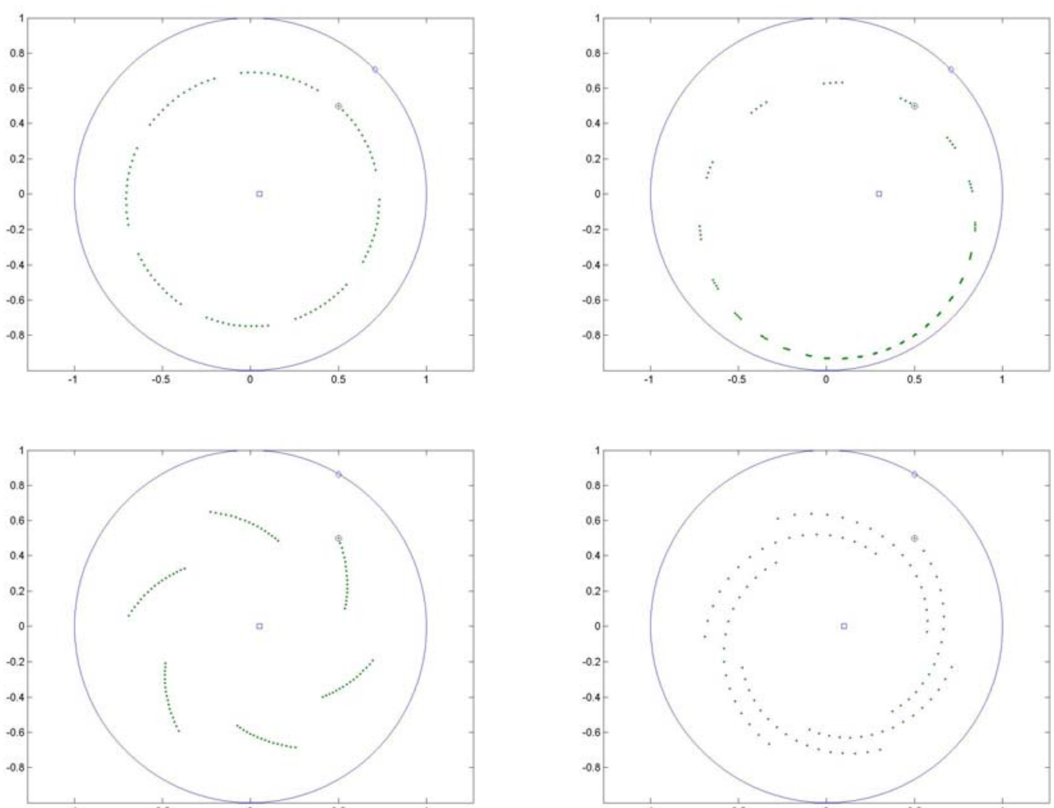


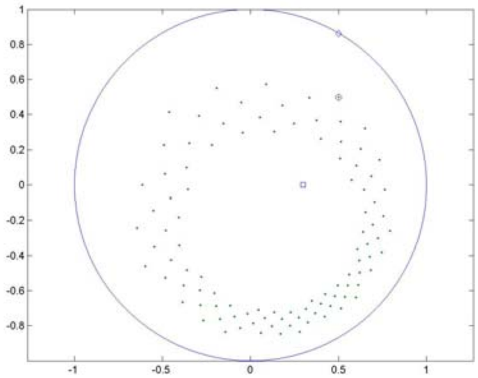
Notice as $a \to 0$, the rate of convergence slows, meaning it takes longer for *T* to converge. This is shown in the last picture on the above-right. We can now do some calculation to see which fixed point a certain picture is converging to.

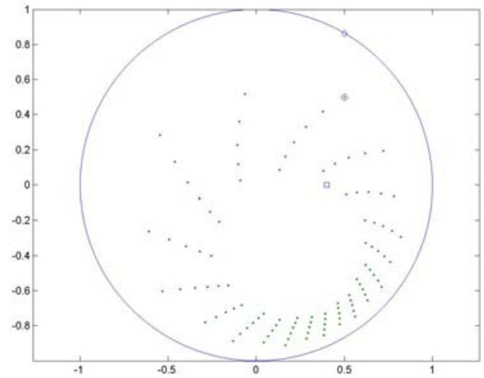
$u = 1$ so $T(z) = \dfrac{z-a}{1-\bar{a}z} = z$, then solve for *z* plugging in the respective value for *a*.

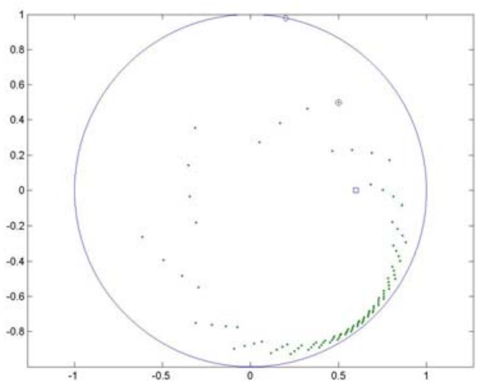
$z = \pm 1$, so let $z_1 = 1, z_2 = -1$. Now find λ. $\lambda = \dfrac{1-(.1)(-1)}{1-(.1)(1)} = 1.1/.9 = 1.2 > 1$. So z_2 is attracting and z_1 is repelling.

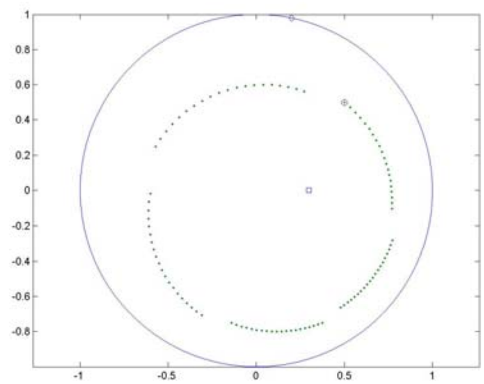
Now here are some pictures when *u* is in other positions (i.e. – elliptic cases):



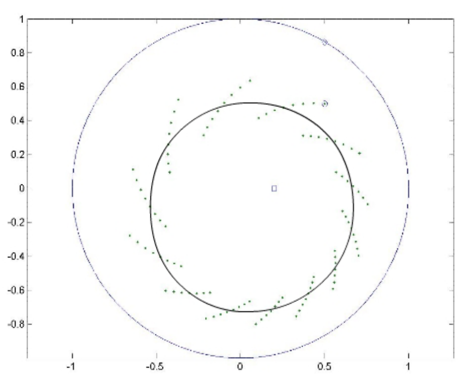


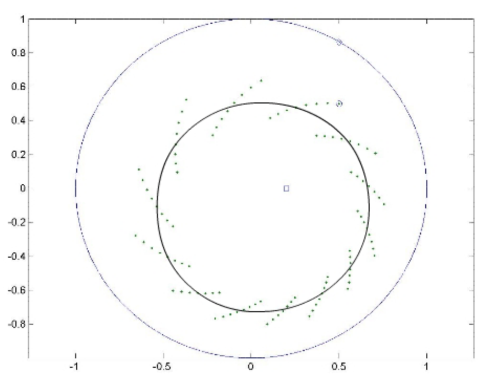


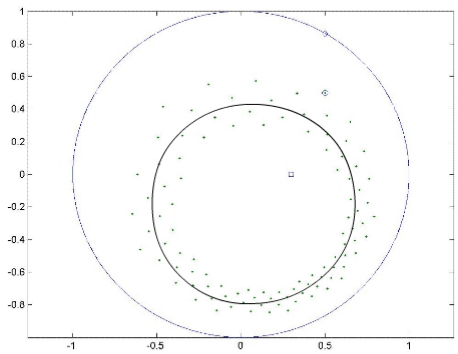
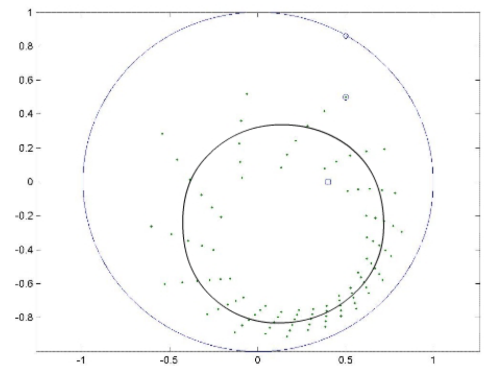




Here are the “average” circle pictures:



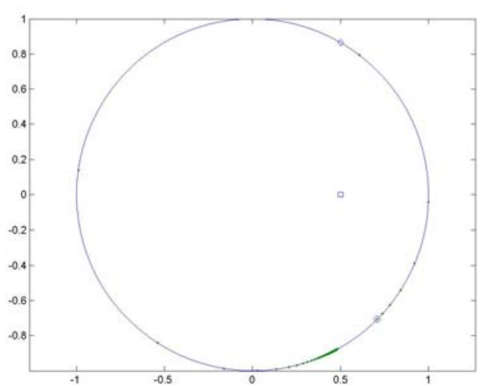
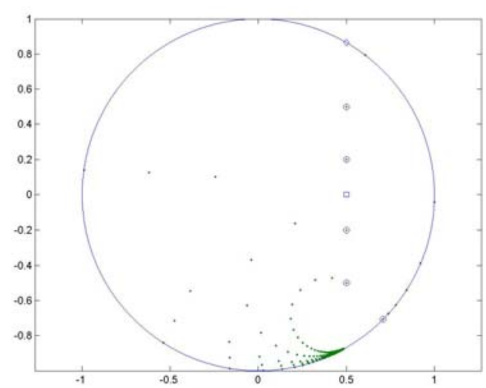


Here is an example of the parabolic case. One needs to find one root of (*), the equation that we found to accomplish this is

$$(1-u)^2 = -4u\,|\,a\,|^2 .$$

This equation has two variables, so we choose one and solve for the other. For this example we chose a = .5, and solved for u. u came out to be $.5+\sqrt{3}/2i$. The fixed point is just the conjugate of u. You can see that in the picture below; notice how in this case that the one fixed point has a repelling and attracting action.

4. APPLICATIONS OF ITERATIVE DYNAMICAL SYSTEMS

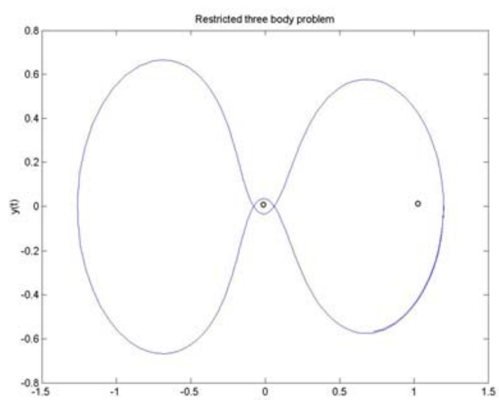
We have basically looked at one kind of transformation that leads to one kind of dynamical system. Here are a few examples of iterative dynamical systems I have previously dealt with before this REU project.

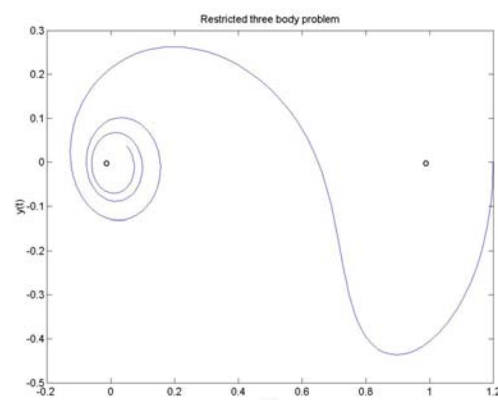
i) *Restricted 3-Body Problem*. The basic idea is that 3 spatial bodies are allowed to rotate by themselves and rotate around each other all gravitationally bound to each other. In full generality this problem is not solvable. So the most popular way to attack this problem is to keep two of the bodies stationary (i.e – a planet / moon system) and allowing the third body to orbit the other two (i.e – a satellite, or a spaceship). One can solve this system using numerical solutions of differential equations. Here are the equations:

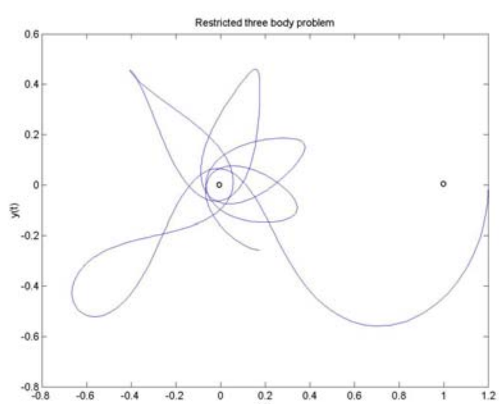
$$y_1' = y_3 \qquad\qquad \mu = 1/82.45$$
$$y_2' = y_4 \qquad\qquad \mu^* = 1-\mu$$
$$y_3' = 2y_4 + y_1 - \frac{\mu^*(y_1+\mu)}{r_1^3} - \frac{\mu(y_1-\mu^*)}{r_2^3} \qquad\qquad r_1 = \sqrt{(y_1+\mu)^2 + y_2^2}$$
$$y_4' = -2y_3 + y_2 - \frac{\mu^* y_2}{r_1^3} - \frac{\mu y_2}{r_2^3} \qquad\qquad r_2 = \sqrt{(y_1-\mu^*)^2 + y_2^2}$$

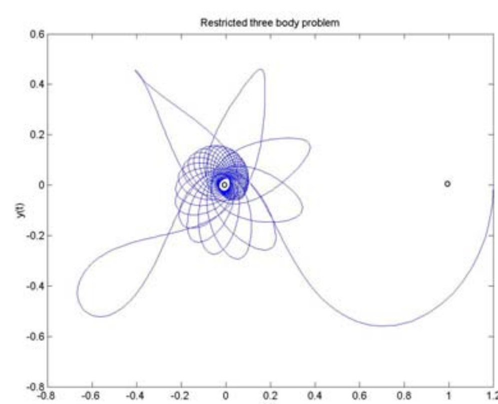
MATLAB was again used to solve this numerical problem.

One can also introduce a “friction” factor *f* (not shown in equations); it physically means the amount of space debris or dust in the system. This first example is where *f* = 0, the second where *f* = 1, and the third is where *f* = .1. Notice this third picture stops after a certain time, the fourth shows it in full. This last example shows how chaotic this system can be.





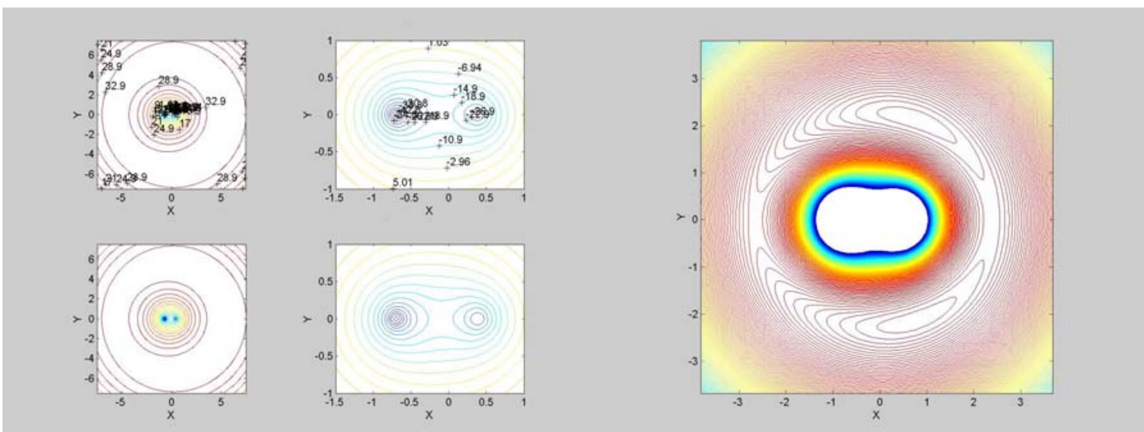




ii) *Binary Galactic system* – This example is simply two galaxies rotating around each other. The reason this is different from a planetary system is that in a planetary system, planets are considered point masses, since a planet is very small on galactic scales. So, we cannot make this assumption when we are dealing with galaxies. The standard equation used for planets is:

$$V(x,y) = \frac{1}{\sqrt{(x-\alpha)^2 + y^2}} - \frac{\alpha}{\sqrt{(x+1-\alpha)^2 + y^2}} - \frac{x^2 + y^2}{2}$$

We altered this potential by substituting different functions for the square root, such as arctangent, and logarithmic. These pictures also show an orbiting object around these two bodies.



The pictures on the left are just a zoomed out image of the one on the right. The colors represent the magnitude of the potential function, the more the red the color is the more energy is required to be in that orbit. The picture on the right shows the many possible orbits of such an orbiting object. This system has many interesting properties but is beyond the scope of this paper.

5. POSSIBLE AREAS OF CONTINUING RESEARCH

One of the areas I would like to study further is the rate of convergence issue. We saw that in the hyperbolic and parabolic cases that the closer *a* was to the orgin the slower the function takes to converge. I would like to see how and why this occurs. In the elliptic case we saw that these "average" circles could be created, it turns out that in the hyperbolic and parabolic cases that we can find "average" arcs of circles. I would like to investigate this further. Also investigating other automorphisms that are not onto (unlike our example) would be interesting as well.

I am very glad to been a part of this program and I thank everyone who took part in it as well as all involved in making the program happen. It is a great experience for all eager, young math students.

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