Polygonal Designs: Existence and Construction

Combinatorics Group: John Hegeman and Jeff Langford
Mentors: G. Bhattacharyya, J. Kim and S. Y. Song

August 24, 2004

Polygonal designs form a special class of partially balanced incomplete block designs. We resolve the existence problem for polygonal designs with various parameter sets and find several construction methods with blocks of small sizes.

For positive integers v, b, k, and r with $2 < k < v$, $1 \leq r < b$, a (doubly regular) *incomplete block design* with parameters (v, b, k, r), is a pair $(V, \mathcal{B})$, where V is a set of v elements, called *points* or *varieties* and $\mathcal{B}$ is a collection of b k-element subsets of V called *blocks*, satisfying the condition: *each point appears in exactly* r *blocks*. An incomplete block design is said to be *balanced* if any two points are contained in exactly λ blocks together. It is called *partially balanced* if every pair of points occurs in a certain number of blocks depending on an association relation between the points.

A special class of partially balanced incomplete block designs called *polygonal designs* can be defined on a regular polygon. The set V forms a v-gon with vertex (point) set

$$V = \mathbb{Z}_v = \{0, 1, 2, \ldots, v-1\}.$$

We define the distance $\delta(x, y)$ between points x and y to be the length of the shortest path connecting x and y on V. That is, $\forall x, y \in V$

$$\delta(x, y) := \min\{|x-y|,\ v-|x-y|\}, \text{ and thus } 0 \leq \delta(x, y) \leq \left\lfloor \frac{v}{2} \right\rfloor .$$

Let $V = \{0, 1, \ldots, v-1\}$ be the point set of a regular v-gon, and let $m < (v-1)/2$. An incomplete block design $(V, \mathcal{B})$ with parameters (v, b, k, r) is called a *polygonal design with minimum interval* m, if any two points of V that are at distance $m+1$ or greater appear together in λ blocks while other pairs do not occur in the blocks at all. This design is denoted by $\mathrm{PD}(v, k, \lambda; m)$. We note that a $\mathrm{PD}(v, k, \lambda; 0)$ is a balanced incomplete block design which is also known as a 2-(v, k, λ) design.

A number of authors have provided solutions to the existence and construction problems of polygonal designs for various combinations of v, b, k and λ. Some of the relevant references, almost all of which deal primarily with the case $m = 1$, are as follows. Hedayat, Rao, and Stufken introduced polygonal designs for the first time in 1988 as balanced sampling plans excluding contiguous units pertaining to finite population sampling. They showed that $v \geq 3k$ is a necessary condition for the existence of PD$(v, k, \lambda; 1)$. They also provided an iterative construction method by showing that if a PD$(v, k, \lambda; 1)$ exists, then a PD$(v + 3\alpha, k, \lambda'; 1)$ exists for any positive integer α. Stufken, Song, See and Driessel (1998) showed that if a PD$(v, k, \lambda; m)$ exists, then $b \geq v$ and $v \geq k(2m + 1)$. They also showed that a PD$(3k, k, \lambda; 1)$ does not exist for any λ if $k \geq 5$. In regard to the construction of designs, Colbourn and Ling (1998, 1999) constructed all PD$(v, k, \lambda; 1)$ for $k = 3$ and $k = 4$. Stufken and Wright (2001) constructed all possible PD$(v, k, \lambda; 1)$ with $k = 5, 6$ and 7, except possibly one, and several designs with block size 9 and 10.

In this project, we have studied the polygonal designs with $v = k(2m + 1)$ for an arbitrary m. We have resolved the existence of polygonal designs with $v = k(2m + 1)$ and $k = 3$ completely. We have shown that if a PD$(k(2m + 1), k, \lambda; m)$ exists, then so does PD$((k - 1)(2m + 1), k - 1, \lambda'; m)$ for some λ'. We have also shown that given a PD$(v, k, \lambda; m)$ there exists PD$(v + (2m + 1)\alpha, k, \lambda'; m)$ for any positive integer α with some λ'. These results are generalization of some of the results provided by Hedayat, Rao and Stufken and Stufken, Song, See and Driessel. We have also provided a new construction method for PD$(v, k, \lambda; m)$ by using a 'perfect (k, m)-grouping'. We have then shown that the inequality $k(k - 1) \leq 4(2m + 1)$ holds in a PD$(k(2m + 1), k, \lambda; m)$. This result confirms the non-existence of PD$(3k, k, \lambda; 1)$ for $k \geq 5$ which was proved by Stufken, Song, See and Driessel.